

On the Triharmonic Lane-Emden Equation*

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Abstract

We derive a monotonicity formula and classify finite Morse index solutions (positive or sign-changing, radial or not) to the following triharmonic Lane-Emden equation:

$$(-\Delta)^3 u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n,$$

where p is below the Joseph-Lundgren exponent. As a byproduct we also obtain a new monotonicity formula for the triharmonic maps.

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1 Introduction and Main results

In this paper, we study the finite Morse index solutions of the following triharmonic Lane-Emden equation

$$(-\Delta)^3 u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n \tag{1.1}$$

and give a complete classification of such kind of solutions.

The Lane-Emden equation

$$-\Delta u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n \tag{1.2}$$

and its parabolic counterpart have played an essential role in the development of methods of nonlinear PDEs in the last decades. A fundamental result on equation (1.2) is the celebrated Liouville-type theorem due to Gidas and Spruck [16]: *The equation (1.2) has no positive classical solution if $0 < p < p_S$, where $p_S := (n+2)/(n-2)$ if $n \geq 3$ while $p_S := \infty$ if $n \leq 2$.* Since then there has been an extensive literature on such a type of equations or systems. In particular, in 2007 the seminar paper [10] by Farina (see also [11]), the equation (1.2) is revisited for $p > \frac{n+2}{n-2}$. The author obtained some classification results and Liouville-type theorems for smooth solutions including stable solutions, finite Morse index solutions, solutions which are stable outside a compact

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set, radial solutions and non-negative solutions. The results obtained in [10] were applied to subcritical, critical and supercritical values of the exponent p . Moreover, the critical stability exponent $p_c(n)$ (Joseph-Lundgren exponent) is determined which is larger than the classical critical exponent $p_S = 2^* - 1$ in Sobolev imbedding theorems. In some sense, the Joseph-Lundgren exponent $p_c(n)$ is a critical threshold for obtaining the Liouville-type theorems for stable or finite Morse index solutions. The proof of Farina involves a delicate use of Nash-Moser's iteration technique, which is a classical tool for regularity of second order elliptic operators and falls short for higher order operators.

The biharmonic Lane-Emden equation:

$$(-\Delta)^2 u = |u|^{p-1} u \quad \text{in } \mathbb{R}^n \quad (1.3)$$

has also attracted lots of studies in recent years. The classical Gidas-Spruck type result has been extended ([21], [27]). The radial solutions are classified ([13], [15]). The classification of stable/finite Morse index solutions was initiated by Cowan-Esposito-Ghoussoub [4] and Cowan-Ghoussoub [5]. A complete classification was obtained by Davila-Dupaigne-Wang-Wei in [7]. They give a complete classification of stable and finite Morse index solutions (whether positive or sign changing), in the full exponent range. To by-pass the Nash-Moser iterations, a key point used in [7] is the monotonicity formula for bi-harmonic equations.

On the other hand, very recently the nonlocal Lane-Emden equation

$$(-\Delta)^s u = |u|^{p-1} u \quad \text{in } \mathbb{R}^n \quad (1.4)$$

were considered in Davila-Dupaigne-Wei [6] when $0 < s < 1$ and Fazly-Wei [18] when $1 < s < 2$. Both [6] and [18] gave a complete classification of finite Morse index solution of (1.4).

The motivation to study the above equations comes from both physics and geometry. In particular, the critical case is inevitable for studying the conformal geometry like the prescribed scalar curvature problem. On the other hand, it is well known that the Liouville-type theorems play a crucial role to get a priori L^∞ -bounds for solutions of semilinear elliptic and parabolic problems. In this regard, we refer to the book by Quittner and Souplet [26].

In this paper we initiate the study of finite Morse index solutions to the triharmonic Lane-Emden equation (1.1). There are three critical exponents. The first one is the Serrin's exponent $\frac{n}{n-6}$. The second is the Sobolev exponent $p_S = \frac{n+6}{n-6}$. The third is the Joseph-Lundgren exponent which is given by the following formula:

$$p_c(n) = \begin{cases} \infty & \text{if } n \leq 14, \\ \frac{n+4-2d(n)}{n-8-2d(n)} & \text{if } n \geq 15, \end{cases} \quad (1.5)$$

where

$$\begin{aligned} d(n) &:= \frac{1}{6} \left(9n^2 + 96 - \frac{1536 + 1152n^2}{d_0(n)} - \frac{3}{2} d_0(n) \right)^{1/2}; \\ d_0(n) &:= -(d_1(n) + 36\sqrt{d_2(n)})^{1/3}; \end{aligned} \quad (1.6)$$

$$\begin{aligned}
d_1(n) &:= -94976 + 20736n + 103104n^2 - 10368n^3 + 1296n^5 - 3024n^4 - 108n^6; \\
d_2(n) &:= 6131712 - 16644096n^2 + 6915840n^4 - 690432n^6 - 3039232n \\
&\quad + 4818944n^3 - 1936384n^5 + 251136n^7 - 30864n^8 - 4320n^9 \\
&\quad + 1800n^{10} - 216n^{11} + 9n^{12}.
\end{aligned}$$

Remark 1.1. In the harmonic case, the Joseph-Lundgren exponent (Joseph-Lundgren [19]) is given by

$$p_{cHarmonic}(n) := \begin{cases} \infty & \text{if } n \leq 10, \\ \frac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n \geq 11 \end{cases} \quad (1.7)$$

while in the bi-harmonic case, the corresponding exponent (Gazzola and Grunau [13]) is

$$p_{cBiharmonic}(n) := \begin{cases} \infty & \text{if } n \leq 12, \\ \frac{n+2-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}}{n-6-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}} & \text{if } n \geq 13. \end{cases} \quad (1.8)$$

In the triharmonic case, $p_c(n)$ satisfies a 6-th order polynomial algebraic equation which in general has no explicit solution. It is interesting that we obtain explicit formula.

Next, we recall several definitions.

Definition 1.1. A solution u of (1.1) is said to be stable if

$$\int_{\mathbb{R}^n} |\nabla \Delta \varphi|^2 dx \geq p \int_{\mathbb{R}^n} |u|^{p-1} \varphi^2 dx, \quad \text{for any } \varphi \in H^3(\mathbb{R}^n).$$

Definition 1.2. A solution u of (1.1) is said to be stable outside a compact $\Theta \subset \mathbb{R}^n$ if

$$\int_{\mathbb{R}^n} |\nabla \Delta \varphi|^2 dx \geq p \int_{\mathbb{R}^n} |u|^{p-1} \varphi^2 dx, \quad \text{for any } \varphi \in H^3(\mathbb{R}^n \setminus \Theta).$$

Definition 1.3. The Morse index of the solution u of (1.1) is defined as the maximal dimension over all subspaces E of $H^3(\mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} |\nabla \Delta \varphi|^2 dx < p \int_{\mathbb{R}^n} |u|^{p-1} \varphi^2 dx, \quad \text{for any } \varphi \in E \setminus \{0\}.$$

Hence, a solution is stable if and only if its Morse index is equal to zero. It is known that if a solution u to (1.1) has finite Morse index, then there exists a compact set $\mathcal{K} \subset \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |\nabla \Delta \varphi|^2 dx \geq p \int_{\mathbb{R}^n} |u|^{p-1} \varphi^2 dx, \quad \text{for any } \varphi \in H^2(\mathbb{R}^n \setminus \mathcal{K}).$$

The first main result of the present paper is the following

Theorem 1.1. *Let u be a stable solution of (1.1). If $1 < p < p_c(n)$, then $u \equiv 0$.*

For finite Morse index solutions we have the following

Theorem 1.2. *Let u be a finite Morse index solution of (1.1). Assume that either*

$$(1) \quad 1 < p < \frac{n+6}{n-6} \text{ or}$$

$$(2) \quad \frac{n+6}{n-6} < p < p_c(n),$$

then the solution $u \equiv 0$.

$$(3) \quad \text{If } p = \frac{n+6}{n-6}, \text{ then } u \text{ has a finite energy, i.e.,}$$

$$\int_{\mathbb{R}^n} |\nabla \Delta u|^2 = \int_{\mathbb{R}^n} |u|^{p+1} < +\infty.$$

Remark 1.2. *In both Theorems the condition $p < p_c(n)$ is optimal. In fact the radial singular solution is stable for $p \geq p_c(n)$. See [22].*

Remark 1.3. *While there are many works on second order and fourth order Lane-Emden equations, there are very few works on 6-th order Lane-Emden equations. We refer to Farina-Ferreio [12], Lazzo-Schmidt [20], Martinazzi [23] and the references therein for related results on polyharmonic nonlinear equations.*

Theorems 1.1-1.2 are proved by Monotonicity Formula which we introduce in the next section.

2 Monotonicity formula for triharmonic Lane-Emden equations

We denote $\partial_r u = \nabla u \cdot \frac{x}{r}$, $r = |x|$. Let $\delta_i, i = 1, 2, 3, 4$ be defined by

$$\begin{aligned} \delta_1 &= 2n - \frac{24}{p-1}, \\ \delta_2 &= n(n-2) - n \frac{36}{p-1} - \frac{36}{p-1} \left(1 + \frac{36}{p-1}\right), \\ \delta_3 &= -\frac{24}{p-1} \left(1 + \frac{6}{p-1}\right) \left(2 + \frac{6}{p-1}\right) + 2n \frac{12}{p-1} \left(1 + \frac{6}{p-1}\right) \\ &\quad - (n+b)(n+b-2) \left(1 + \frac{12}{p-1}\right), \\ \delta_4 &= \left(3 + \frac{6}{p-1}\right) \left(2 + \frac{6}{p-1}\right) \left(1 + \frac{6}{p-1}\right) \frac{6}{p-1} - 2n \left(1 + \frac{6}{p-1}\right) \left(2 + \frac{6}{p-1}\right) \frac{6}{p-1} \\ &\quad + n(n-2) \left(2 + \frac{6}{p-1}\right) \frac{6}{p-1}. \end{aligned} \tag{2.1}$$

Next, we will introduce a functional and consider its monotonicity formula. Let

$$B_\lambda := \{y \in \mathbb{R}^n : |y - x| < \lambda\}, \quad \lambda > 0.$$

We define the functional $E(\lambda, x, u)$ depending on $x \in \mathbb{R}^n, \lambda > 0$ and u :

$$\begin{aligned}
& E(\lambda, x, u) \\
&:= \lambda^{6\frac{p+1}{p-1}-n} \left(\int_{B_\lambda} \frac{1}{2} |\nabla \Delta u|^2 - \frac{1}{p+1} \int_{B_\lambda} |u|^{p+1} \right) \\
&\quad - \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} \left[\frac{6}{p-1} \left(\frac{6}{p-1} - 1 \right) \left(\frac{6}{p-1} - 2 \right) \left(\frac{6}{p-1} - 3 \right) u \right. \\
&\quad + \frac{24}{p-1} \left(\frac{6}{p-1} - 1 \right) \left(\frac{6}{p-1} - 2 \right) \lambda \partial_r u + \frac{36}{p-1} \left(\frac{6}{p-1} - 1 \right) \lambda^2 \partial_{rr} u \\
&\quad + \frac{24}{p-1} \lambda^3 \partial_{rrr} u + \lambda^4 \partial_{rrrr} u \left. \right] \left[\frac{6}{p-1} u + r \lambda \partial_r u \right] \\
&\quad + 2 \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} \left(\frac{6}{p-1} \left(\frac{6}{p-1} - 1 \right) \left(\frac{6}{p-1} - 2 \right) \left(\frac{6}{p-1} - 3 \right) u \right. \\
&\quad + \frac{24}{p-1} \left(\frac{6}{p-1} - 1 \right) \left(\frac{6}{p-1} - 2 \right) \lambda \partial_r u + \frac{36}{p-1} \left(\frac{6}{p-1} - 1 \right) \lambda^2 \partial_{rr} u \\
&\quad + \frac{24}{p-1} \lambda^3 \partial_{rrr} u + \lambda^4 \partial_{rrrr} u \left. \right) \left(\frac{6}{p-1} u + r \lambda \partial_r u \right) \\
&\quad - (\delta_1 - 6) \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} \left[\frac{6}{p-1} \left(\frac{6}{p-1} - 1 \right) \left(\frac{6}{p-1} - 2 \right) u \right. \\
&\quad + \frac{18}{p-1} \left(\frac{6}{p-1} - 1 \right) \lambda \partial_r u + \frac{18}{p-1} \lambda^2 \partial_{rr} u + \lambda^3 \partial_{rrr} u \left. \right] \\
&\quad \left[\frac{6}{p-1} u + \lambda \partial_r u \right] + \left(\frac{6}{p-1} + 2 \right) \lambda^{6\frac{p+1}{p-1}-n-1} \int_{\partial B_\lambda} (\Delta_b u)^2 \\
&\quad - (24 - 6\delta_1 + \delta_2) \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} \left[\frac{6}{p-1} \left(\frac{6}{p-1} - 1 \right) u \right. \\
&\quad + \frac{14}{p-1} \lambda \partial_r u + \lambda^2 \partial_{rr} u \left. \right] \left[\frac{6}{p-1} u + \lambda \partial_r u \right] \\
&\quad - (9\delta_1 - 3\delta_2 - 36) \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} \left[\frac{6}{p-1} u + \lambda \partial_r u \right]^2 \\
&\quad + (\delta_1 - 8) \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} \left[\frac{6}{p-1} \left(\frac{6}{p-1} - 1 \right) u \right. \\
&\quad + \frac{12}{p-1} \lambda \partial_r u + \lambda^2 \partial_{rr} u \left. \right]^2 \\
&\quad + \delta_4 \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} \left(\frac{6}{p-1} u + \lambda \partial_r u \right) u
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
& + 2\delta_4 \int_{\partial B_\lambda} \lambda^{2s\frac{p+1}{p-1}-n-5} u^2 \\
& + 2 \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} (\lambda^2 \Delta u - \lambda^2 \partial_{rr} u - (n-1)\lambda \partial_r u)^2 \\
& + \frac{1}{2} \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-4} \frac{d}{d\lambda} (\lambda^2 \Delta u - \lambda^2 \partial_{rr} u - (n-1)\lambda \partial_r u)^2 \\
& - 4(\beta - n + 3) \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} [\lambda^2 |\nabla u|^2 - \lambda^2 |\partial_r u|^2] \\
& - (\beta - n + 3) \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-4} \frac{d}{d\lambda} [\lambda^2 |\nabla u|^2 - \lambda^2 |\partial_r u|^2] \\
& - 6 \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-4} \frac{d}{d\lambda} [\lambda^2 |\nabla(\frac{6}{p-1}u + \lambda \partial_r u)|^2 - \lambda^2 |\partial_r(\frac{6}{p-1}u + \lambda \partial_r u)|^2] \\
& + \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-2} \frac{d}{d\lambda} [|\nabla(\frac{6}{p-1}u + \lambda \partial_r u)|^2 - |\partial_r(\frac{6}{p-1}u + \lambda \partial_r u)|^2]^2,
\end{aligned}$$

where $\delta_j (j = 1, 2, 3, 4)$ are defined in (2.1),

The following is the monotonicity formula which will play an important role.

Theorem 2.1. *Let u satisfy the equation (1.1). Define $u^\lambda(x) = \lambda^{\frac{6}{p-1}} u(\lambda x)$, then*

$$\begin{aligned}
& \frac{dE(\lambda, x, u)}{d\lambda} \\
& = \int_{\partial B_1} \left(2\lambda^5 \left(\frac{d^3 u^\lambda}{d\lambda^3} \right)^2 + (10\delta_1 - 2\delta_2 - 56)\lambda^3 \left(\frac{d^2 u^\lambda}{d\lambda^2} \right)^2 \right. \\
& \quad \left. + (-18\delta_1 + 6\delta_2 - 4\delta_3 + 2\delta_4 + 72)\lambda \left(\frac{du^\lambda}{d\lambda} \right)^2 \right) \\
& \quad + \int_{\partial B_1} \left(4\lambda^3 \left(\frac{d^2}{d\lambda^2} \nabla_\theta u^\lambda \right)^2 + (8\alpha - 4\beta + 4n - 28)\lambda \left(\frac{d}{d\lambda} \nabla_\theta u^\lambda \right)^2 \right) \\
& \quad + 2\lambda \int_{\partial B_1} \left[\frac{d}{d\lambda} \mathbf{div}_\theta(\nabla_\theta u^\lambda) \right]^2 + \lambda \int_{\partial B_1} \left(\frac{d\Delta u^\lambda}{d\lambda} \right)^2.
\end{aligned} \tag{2.3}$$

We will give the proof of Theorem 2.1 in the next section. Now we would like to state a consequence of Theorem 2.1.

The functional $E(\lambda, x, u)$, defined in (2.2), can be divided into two parts: the integral over the ball B_λ and the terms of integrals on the boundary ∂B_λ . We notice that in the blow-down analysis process, the boundary terms can be controlled using initial energy estimates. Then, we may change some coefficients of the boundary terms in $E(\lambda, x, u)$, we denote it by $E^c(\lambda, x, u)$, which may be formulated in the following way:

$$\begin{aligned}
& E^c(\lambda, x, u) \\
& := E(\lambda, x, u) - \int_{\partial B_1} \left(\sum_{0 \leq i, j \leq 2} c_{i,j} \lambda^{i+j} \frac{d^i u^\lambda}{d\lambda^i} \frac{d^j u^\lambda}{d\lambda^j} \right),
\end{aligned} \tag{2.4}$$

where $c_{i,j} \in \mathbb{R}$ are chosen properly which may be different by various cases. Moreover, we still can obtain the lower bound of $\frac{dE^c(\lambda, x, u)}{d\lambda}$ and the lower bound is independent of $c_{i,j} \in \mathbb{R}$. We have the following precise statement

Theorem 2.2. *Assume that $\frac{n+6}{n-6} < p < p_m(n)$. Then there exist c_{ij} such that $E^c(\lambda, x, u)$, defined at (2.4), is a nondecreasing function of $\lambda > 0$. Furthermore,*

$$\frac{dE^c(\lambda, x, u)}{d\lambda} \geq C(n, p) \lambda^{6\frac{p+1}{p-1}-6-n} \int_{\partial B_\lambda(x_0)} \left(\frac{6}{p-1} u + \lambda \partial_r u \right)^2, \quad (2.5)$$

where $C(n, p) > 0$ is a constant independent of λ , and

$$p_m(n) := \begin{cases} +\infty & \text{if } n \leq 30, \\ \frac{5n+30-\sqrt{15n^2-60n+190}}{5n-30-\sqrt{15n^2-60n+190}} & \text{if } n \geq 31. \end{cases}$$

Remark 2.1. *In the above theorem, we need the upper bound condition of p , namely $p < p_m(n)$. Let us recall that in the biharmonic case the monotonicity formula holds for all $p > \frac{n+4}{n-4}$ (see [7]). Since $p_c(n) < p_m(n)$, the above monotonicity formula holds for $\frac{n+6}{n-6} < p < p_c(n)$ which is used for our blow down analysis. See Theorem 7.1. It seems that in the triharmonic case, the supercritical condition $p > \frac{n+6}{n-6}$ alone is not sufficient to make such kind of monotonicity formula (2.6) hold. We refer the readers to section 7 and [1] for more details.*

Remark 2.2. *The proof of Theorem 2.2 is quite involved. In the bi-harmonic cases, the positivity of $\frac{dE}{d\lambda}$ is trivial. Here we have to discuss three cases: $n \leq 20$, $21 \leq n \leq 30$ and $n \geq 31$. In each case we have to come up with different combinations of terms.*

Remark 2.3. *In [1], Simon Blatt also derived a monotonicity formula for triharmonic Lane-Emden equations under different conditions on p which is much stronger than our's here (see below). He then used to prove partial regularity of stationary solutions and obtain Hausdorff dimension estimates for the singular set of solutions. By unifying the notations, we see that in [1] (Corollary 3.13), the author gets the monotonicity under the condition $-(n-20) + 8\alpha(n-1) \geq 0$. Transfer to our notations, the monotonicity formula of [1] requires the hypothesis $\frac{n+6}{n-6} < p < p_{m_1}(n)$, where*

$$p_{m_1}(n) := \begin{cases} +\infty & \text{if } n \leq 20, \\ \frac{n+28}{n-20} & \text{if } n \geq 21. \end{cases}$$

A direct calculation shows that $p_{m_1}(n) < p_m(n)$. Therefore, by using our arguments in the current paper, the second main result Theorem 1.2 of [1] actually can be improved to $\frac{n+6}{n-6} < p < p_m(n)$.

By slightly modifying the proof of Theorem 2.2, we are able to get the monotonicity formula for the triharmonic map, i.e.,

$$\Delta^3 u = 0.$$

Indeed, let $p \rightarrow +\infty$ in (2.2) and denote $E_\infty(\lambda, x, u) = \lim_{p \rightarrow \infty} E(\lambda, x, u)$, where the term $\frac{1}{p+1} \lambda^{6\frac{p+1}{p-1}-n} \int_{\partial B_\lambda} |u^\lambda|^{p+1}$ is understood vanished, then we have

Corollary 2.1. *Assume that $7 \leq n \leq 30$. Then there exist c_{ij} such that $E_\infty^c(\lambda, x, u)$, defined similarly as in (2.4), is a nondecreasing function of $\lambda > 0$. Furthermore,*

$$\frac{dE_\infty^c(\lambda, x, u)}{d\lambda} \geq C(n)\lambda^{-n} \int_{\partial B_\lambda(x_0)} \left(\lambda \partial_r u \right)^2, \quad (2.6)$$

where $C(n) > 0$ is a constant independent of λ .

Remark 2.4. *In [1], Simon Blatt derived a monotonicity formula for extrinsic triharmonic maps under the conditions on $6 < n \leq 20$ by which a smoothness result was obtained (See Theorem 1.1 of [1]). However, by our results here, his Theorem 1.1 can be improved to $6 < n \leq 30$.*

At the end of this section, we say a few words on the powerful applications of monotonicity formula. It is known that monotonicity formulas are one of the most important tools for studying geometric problems as well as supercritical equations and systems. For monotonicity formulas for stationary harmonic maps we refer to Evans [8] for harmonic maps and Chang-Wang-Yang [9] for biharmonic maps. For the second order Lane-Emden equation we refer to Giga-Kohn [14] and Pacard [24]. For biharmonic and fractional Lane-Emden equations we refer to [6, 7, 18].

3 Monotonicity formula and the proof of Theorem 2.1

Since the derivation of the derivative for the $E(\lambda, x, u)$ is complicated, we divide it into several subsections. In subsection 3.1, we derive $\frac{d}{d\lambda} \overline{E}(u, \lambda)$, where $\overline{E}(u, \lambda)$ is defined in (3.1) below, which in fact is the first term of $E(\lambda, x, u)$ introduced in (2.2). In subsection 3.2, we calculate the (higher-order) derivatives $\frac{\partial^j}{\partial r^j} u^\lambda$ and $\frac{\partial^i}{\partial \lambda^i} u^\lambda$, $i, j = 1, 2, 3, 4$. In subsection 3.3, the operator Δ^2 and its representation will be given. In subsection 3.4, we decompose $\frac{d}{d\lambda} \overline{E}(u^\lambda, 1)$. Finally, combining with the above four subsections, we can obtain the derivative formula, hence get the proof of Theorem 2.1.

Without loss of generality, suppose that $x_0 = 0$ and denote by B_λ the ball centered at zero with radius λ . Set

$$\overline{E}(u, \lambda) := \lambda^{6\frac{p+1}{p-1}-n} \left(\int_{B_\lambda} \frac{1}{2} |\nabla \Delta u|^2 - \frac{1}{p+1} \int_{B_\lambda} |u|^{p+1} \right). \quad (3.1)$$

3.1 The derivation of $\frac{d}{d\lambda} \overline{E}(u, \lambda)$

Define

$$\begin{aligned} v &:= \Delta u, & u^\lambda(x) &:= \lambda^{\frac{6}{p-1}} u(\lambda x), & w &:= \Delta v, \\ v^\lambda(x) &:= \lambda^{\frac{6}{p-1}+2} v(\lambda x), & w^\lambda(x) &:= \lambda^{\frac{6}{p-1}+4} w(\lambda x). \end{aligned} \quad (3.2)$$

Therefore,

$$\Delta u^\lambda(x) = v^\lambda(x), \quad \Delta v^\lambda(x) = w^\lambda(x). \quad (3.3)$$

In addition, differentiating (3.3) with respect to λ we have

$$\Delta \frac{du^\lambda}{d\lambda} = \frac{dv^\lambda}{d\lambda}, \quad \Delta \frac{dv^\lambda}{d\lambda} = \frac{dw^\lambda}{d\lambda}.$$

Note that

$$\overline{E}(u, \lambda) = \overline{E}(u^\lambda, 1) = \int_{B_1} \frac{1}{2} |\nabla v^\lambda|^2 - \frac{1}{p+1} \int_{B_1} |u^\lambda|^{p+1}.$$

Taking derivative of the energy $\overline{E}(u^\lambda, 1)$ with respect to λ and integrating by part, we have :

$$\begin{aligned} \frac{d\overline{E}(u^\lambda, 1)}{d\lambda} &= \int_{B_1} \nabla v^\lambda \nabla \frac{dv^\lambda}{d\lambda} - \int_{B_1} |u^\lambda|^{p-1} u^\lambda \frac{du^\lambda}{d\lambda} \\ &= \int_{B_1} \nabla v^\lambda \nabla \frac{dv^\lambda}{d\lambda} + \int_{B_1} \Delta w^\lambda \frac{du^\lambda}{d\lambda}. \end{aligned} \quad (3.4)$$

Next, we calculate the term $\int_{B_1} \nabla v^\lambda \nabla \frac{dv^\lambda}{d\lambda}$:

$$\begin{aligned} \int_{B_1} \nabla v^\lambda \nabla \frac{dv^\lambda}{d\lambda} &= \int_{\partial B_1} \frac{\partial v^\lambda}{\partial r} \frac{dv^\lambda}{d\lambda} - \int_{B_1} \Delta v^\lambda \frac{dv^\lambda}{d\lambda} \\ &= \int_{\partial B_1} \frac{\partial v^\lambda}{\partial r} \frac{dv^\lambda}{d\lambda} - \int_{B_1} w^\lambda \Delta \frac{du^\lambda}{d\lambda} \\ &= \int_{\partial B_1} \frac{\partial v^\lambda}{\partial r} \frac{dv^\lambda}{d\lambda} - \int_{\partial B_1} w^\lambda \frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda} + \int_{B_1} \nabla w^\lambda \nabla \frac{du^\lambda}{d\lambda} \\ &= \int_{\partial B_1} \frac{\partial v^\lambda}{\partial r} \frac{dv^\lambda}{d\lambda} - \int_{\partial B_1} w^\lambda \frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda} \\ &\quad + \int_{\partial B_1} \frac{\partial w^\lambda}{\partial r} \frac{du^\lambda}{d\lambda} - \int_{B_1} \Delta w^\lambda \frac{du^\lambda}{d\lambda}. \end{aligned} \quad (3.5)$$

By (3.4) and (3.5) we obtain that

$$\frac{d}{d\lambda} \overline{E}(u^\lambda, 1) = \int_{\partial B_1} \frac{\partial v^\lambda}{\partial r} \frac{dv^\lambda}{d\lambda} + \int_{\partial B_1} \frac{\partial w^\lambda}{\partial r} \frac{du^\lambda}{d\lambda} - \int_{\partial B_1} w^\lambda \frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda}. \quad (3.6)$$

Recalling (3.2) and differentiating it with respect to λ , we have

$$\begin{aligned} \frac{du^\lambda(x)}{d\lambda} &= \frac{1}{\lambda} \left(\frac{6}{p-1} u^\lambda(x) + r \partial_r u^\lambda(x) \right), \\ \frac{dv^\lambda(x)}{d\lambda} &= \frac{1}{\lambda} \left(\left(\frac{6}{p-1} + 2 \right) v^\lambda(x) + r \partial_r v^\lambda(x) \right), \\ \frac{dw^\lambda(x)}{d\lambda} &= \frac{1}{\lambda} \left(\left(\frac{6}{p-1} + 4 \right) w^\lambda(x) + r \partial_r w^\lambda(x) \right). \end{aligned}$$

Differentiating the above equations with respect to λ again we get

$$\lambda \frac{d^2 u^\lambda(x)}{d\lambda^2} + \frac{du^\lambda(x)}{d\lambda} = \frac{6}{p-1} \frac{du^\lambda(x)}{d\lambda} + r \partial_r \frac{du^\lambda}{d\lambda}.$$

Hence, for $x \in B_1$, we have

$$\begin{aligned}\partial_r(u^\lambda(x)) &= \lambda \frac{du^\lambda}{d\lambda} - \frac{6}{p-1} u, \\ \partial_r\left(\frac{du^\lambda(x)}{d\lambda}\right) &= \lambda \frac{d^2 u^\lambda(x)}{d\lambda^2} + \left(1 - \frac{6}{p-1}\right) \frac{du^\lambda}{d\lambda}, \\ \partial_r(v^\lambda(x)) &= \lambda \frac{dv^\lambda}{d\lambda} - \left(\frac{6}{p-1} + 2\right) v^\lambda, \\ \partial_r(w^\lambda(x)) &= \lambda \frac{dw^\lambda}{d\lambda} - \left(\frac{6}{p-1} + 4\right) w^\lambda.\end{aligned}$$

Plugging these equations into (3.6), we get that

$$\begin{aligned}\frac{d}{d\lambda} \overline{E}(u^\lambda, 1) &= \int_{\partial B_1} \left(\lambda \frac{dv^\lambda}{d\lambda} \frac{dv^\lambda}{d\lambda} - \left(\frac{6}{p-1} + 2\right) v^\lambda \frac{dv^\lambda}{d\lambda} \right) \\ &\quad + \left(\lambda \frac{dw^\lambda}{d\lambda} \frac{u^\lambda}{d\lambda} - \left(\frac{6}{p-1} + 4\right) w^\lambda \frac{du^\lambda}{d\lambda} \right) \\ &\quad - \left(\lambda w^\lambda \frac{d^2 u^\lambda}{d\lambda^2} + \left(1 - \frac{6}{p-1}\right) w^\lambda \frac{du^\lambda}{d\lambda} \right) \\ &= \int_{\partial B_1} \underbrace{\left[\lambda \frac{dv^\lambda}{d\lambda} \frac{dv^\lambda}{d\lambda} - \left(\frac{6}{p-1} + 2\right) v^\lambda \frac{dv^\lambda}{d\lambda} \right]}_{\text{Term 1}} \\ &\quad + \underbrace{\left[\lambda \frac{dw^\lambda}{d\lambda} \frac{du^\lambda}{d\lambda} - \lambda w^\lambda \frac{d^2 u^\lambda}{d\lambda^2} \right]}_{\text{Term 2}} - 5 w^\lambda \frac{du^\lambda}{d\lambda} \\ &:= \overline{E}_{d1}(u^\lambda, 1) + \overline{E}_{d2}(u^\lambda, 1).\end{aligned}\tag{3.7}$$

3.2 The calculations of $\frac{\partial^j}{\partial r^j} u^\lambda$ and $\frac{\partial^i}{\partial \lambda^i} u^\lambda$, $i, j = 1, 2, 3, 4$

Note

$$\lambda \frac{du^\lambda}{d\lambda} = \frac{6}{p-1} u^\lambda + r \frac{\partial}{\partial r} u^\lambda.\tag{3.8}$$

Differentiating (3.8) once, twice and thrice with respect to λ respectively, we have

$$\lambda \frac{d^2 u^\lambda}{d\lambda^2} + \frac{du^\lambda}{d\lambda} = \frac{6}{p-1} \frac{du^\lambda}{d\lambda} + r \frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda},\tag{3.9}$$

$$\lambda \frac{d^3 u^\lambda}{d\lambda^3} + 2 \frac{d^2 u^\lambda}{d\lambda^2} = \frac{6}{p-1} \frac{d^2 u^\lambda}{d\lambda^2} + r \frac{\partial}{\partial r} \frac{d^2 u^\lambda}{d\lambda^2},\tag{3.10}$$

$$\lambda \frac{d^4 u^\lambda}{d\lambda^4} + 3 \frac{d^3 u^\lambda}{d\lambda^3} = \frac{6}{p-1} \frac{d^3 u^\lambda}{d\lambda^3} + r \frac{\partial}{\partial r} \frac{d^3 u^\lambda}{d\lambda^3}.\tag{3.11}$$

Similarly, differentiating (3.8) once, twice and thrice with respect to r respectively we have

$$\lambda \frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda} = \left(\frac{6}{p-1} + 1\right) \frac{\partial}{\partial r} u^\lambda + r \frac{\partial^2}{\partial r^2} u^\lambda, \quad (3.12)$$

$$\lambda \frac{\partial^2}{\partial r^2} \frac{du^\lambda}{d\lambda} = \left(\frac{6}{p-1} + 2\right) \frac{\partial^2}{\partial r^2} u^\lambda + r \frac{\partial^3}{\partial r^3} u^\lambda, \quad (3.13)$$

$$\lambda \frac{\partial^3}{\partial r^3} \frac{du^\lambda}{d\lambda} = \left(\frac{6}{p-1} + 3\right) \frac{\partial^3}{\partial r^3} u^\lambda + r \frac{\partial^4}{\partial r^4} u^\lambda. \quad (3.14)$$

From (3.8), on ∂B_1 , we have

$$\frac{\partial u^\lambda}{\partial r} = \lambda \frac{du^\lambda}{d\lambda} - \frac{6}{p-1} u^\lambda.$$

Next from (3.9), on ∂B_1 , we derive that

$$\frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda} = \lambda \frac{d^2 u^\lambda}{d\lambda^2} + \left(1 - \frac{6}{p-1}\right) \frac{du^\lambda}{d\lambda}.$$

From (3.12), combining the two equations above, on ∂B_1 , we get

$$\begin{aligned} \frac{\partial^2}{\partial r^2} u^\lambda &= \lambda \frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda} - \left(1 + \frac{6}{p-1}\right) \frac{\partial}{\partial r} u^\lambda \\ &= \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} - \lambda \frac{12}{p-1} \frac{du^\lambda}{d\lambda} + \left(1 + \frac{6}{p-1}\right) \frac{6}{p-1} u^\lambda. \end{aligned} \quad (3.15)$$

Differentiating (3.9) with respect to r , and combine with (3.9) and (3.10), we get that

$$\begin{aligned} \frac{\partial^2}{\partial r^2} \frac{du^\lambda}{d\lambda} &= \lambda \frac{\partial}{\partial r} \frac{d^2 u^\lambda}{d\lambda^2} - \frac{6}{p-1} \frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda} \\ &= \lambda^2 \frac{d^3 u^\lambda}{d\lambda^3} + \left(2 - \frac{12}{p-1}\right) \lambda \frac{d^2 u^\lambda}{d\lambda^2} - \left(1 - \frac{6}{p-1}\right) \frac{6}{p-1} \frac{du^\lambda}{d\lambda}. \end{aligned} \quad (3.16)$$

From (3.13), on ∂B_1 , combine with (3.15) and (3.16), we have

$$\begin{aligned} \frac{\partial^3}{\partial r^3} u^\lambda &= \lambda \frac{\partial^2}{\partial r^2} \frac{du^\lambda}{d\lambda} - \left(2 + \frac{6}{p-1}\right) \frac{\partial^2}{\partial r^2} u^\lambda \\ &= \lambda^3 \frac{d^3 u^\lambda}{d\lambda^3} - \lambda^2 \frac{18}{p-1} \frac{d^2 u^\lambda}{d\lambda^2} + \lambda \left(\frac{18}{p-1} + \frac{108}{(p-1)^2}\right) \frac{du^\lambda}{d\lambda} \\ &\quad - \left(2 + \frac{6}{p-1}\right) \left(1 + \frac{6}{p-1}\right) \frac{6}{p-1} u^\lambda. \end{aligned} \quad (3.17)$$

Now differentiating (3.9) once with respect to r , we get

$$\lambda \frac{\partial^2}{\partial r^2} \frac{d^2 u^\lambda}{d\lambda^2} = \left(\frac{6}{p-1} + 1\right) \frac{\partial^2}{\partial r^2} \frac{du^\lambda}{d\lambda} + r \frac{\partial^3}{\partial r^3} \frac{du^\lambda}{d\lambda},$$

then on ∂B_1 , we have

$$\frac{\partial^3}{\partial r^3} \frac{du^\lambda}{d\lambda} = \lambda \frac{\partial^2}{\partial r^2} \frac{d^2 u^\lambda}{d\lambda^2} - \left(\frac{6}{p-1} + 1\right) \frac{\partial^2}{\partial r^2} \frac{du^\lambda}{d\lambda}. \quad (3.18)$$

Now differentiating (3.10) twice with respect to r , we get

$$\lambda \frac{\partial}{\partial r} \frac{d^3 u^\lambda}{d\lambda^3} = \left(\frac{6}{p-1} - 1\right) \frac{\partial}{\partial r} \frac{d^2 u^\lambda}{d\lambda^2} + r \frac{\partial^2}{\partial r^2} \frac{d^2 u^\lambda}{d\lambda^2}.$$

Hence on ∂B_1 , combining with (3.10) and (3.11) there holds

$$\begin{aligned} \frac{\partial^2}{\partial r^2} \frac{d^2 u^\lambda}{d\lambda^2} &= \lambda \frac{\partial}{\partial r} \frac{d^3 u^\lambda}{d\lambda^3} + \left(1 - \frac{6}{p-1}\right) \frac{\partial}{\partial r} \frac{d^2 u^\lambda}{d\lambda^2} \\ &= \lambda^2 \frac{d^4 u^\lambda}{d\lambda^4} + \lambda \left(4 - \frac{12}{p-1}\right) \frac{d^3 u^\lambda}{d\lambda^3} + \left(1 - \frac{6}{p-1}\right) \left(2 - \frac{6}{p-1}\right) \frac{d^2 u^\lambda}{d\lambda^2}. \end{aligned} \quad (3.19)$$

Now differentiating (3.9) with respect to r , we have

$$\lambda \frac{\partial}{\partial r} \frac{d^2 u^\lambda}{d\lambda^2} = \frac{6}{p-1} \frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda} + r \frac{\partial^2}{\partial r^2} \frac{du^\lambda}{d\lambda}.$$

From this combined with (3.9) and (3.10), on ∂B_1 , we have

$$\begin{aligned} \frac{\partial^2}{\partial r^2} \frac{du^\lambda}{d\lambda} &= \lambda \frac{\partial}{\partial r} \frac{d^2 u^\lambda}{d\lambda^2} - \frac{6}{p-1} \frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda} \\ &= \lambda^2 \frac{d^3 u^\lambda}{d\lambda^3} + \lambda \left(2 - \frac{12}{p-1}\right) \frac{d^2 u^\lambda}{d\lambda^2} - \frac{6}{p-1} \left(1 - \frac{6}{p-1}\right) \frac{du^\lambda}{d\lambda}. \end{aligned} \quad (3.20)$$

Now from (3.18), combining with (3.19) and (3.20), we get

$$\begin{aligned} \frac{\partial^3}{\partial r^3} \frac{du^\lambda}{d\lambda} &= \lambda^3 \frac{d^4 u^\lambda}{d\lambda^4} + \lambda^2 \left(3 - \frac{18}{p-1}\right) \frac{d^3 u^\lambda}{d\lambda^3} - \lambda \left(1 - \frac{6}{p-1}\right) \frac{18}{p-1} \frac{d^2 u^\lambda}{d\lambda^2} \\ &\quad + \left(1 - \frac{6}{p-1}\right) \left(1 + \frac{6}{p-1}\right) \frac{6}{p-1} \frac{du^\lambda}{d\lambda}. \end{aligned} \quad (3.21)$$

From (3.14), on ∂B_1 , combining with (3.21) yields

$$\begin{aligned} \frac{\partial^4}{\partial r^4} u^\lambda &= \lambda \frac{\partial^3}{\partial r^3} \frac{du^\lambda}{d\lambda} - \left(3 + \frac{6}{p-1}\right) \frac{\partial^3}{\partial r^3} u^\lambda \\ &= \lambda^4 \frac{d^4 u^\lambda}{d\lambda^4} - \lambda^3 \frac{24}{p-1} \frac{d^3 u^\lambda}{d\lambda^3} + \lambda^2 \left(2 + \frac{12}{p-1}\right) \frac{18}{p-1} \frac{d^2 u^\lambda}{d\lambda^2} \\ &\quad - \lambda \left(1 + \frac{6}{p-1}\right) \left(1 + \frac{3}{p-1}\right) \frac{48}{p-1} \frac{du^\lambda}{d\lambda} \\ &\quad + \left(3 + \frac{6}{p-1}\right) \left(2 + \frac{6}{p-1}\right) \left(1 + \frac{6}{p-1}\right) \frac{6}{p-1} u^\lambda. \end{aligned}$$

In summary, we have that

$$\begin{aligned}\frac{\partial^3}{\partial r^3}u^\lambda &= \lambda^3 \frac{d^3 u^\lambda}{d\lambda^3} - \lambda^2 \frac{18}{p-1} \frac{d^2 u^\lambda}{d\lambda^2} + \lambda \left(\frac{18}{p-1} + \frac{108}{(p-1)^2} \right) \frac{du^\lambda}{d\lambda} \\ &\quad - \left(2 + \frac{6}{p-1} \right) \left(1 + \frac{6}{p-1} \right) \frac{6}{p-1} u^\lambda\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial r^2}u^\lambda &= \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} - \lambda \frac{12}{p-1} \frac{du^\lambda}{d\lambda} + \left(1 + \frac{6}{p-1} \right) \frac{6}{p-1} u^\lambda \\ \frac{\partial u^\lambda}{\partial r} &= \lambda \frac{du^\lambda}{d\lambda} - \frac{6}{p-1} u^\lambda.\end{aligned}$$

3.3 On the operator Δ^2 and its representation

Note that

$$\Delta u = \nabla \cdot (\nabla u) = u_{rr} + \frac{n-1}{r} u_r + \frac{1}{r^2} \operatorname{div}_\theta(\nabla_\theta u).$$

Set $v := \Delta u$ and $w := \Delta^2 u$. Then

$$\begin{aligned}w = \Delta v &= v_{rr} + \frac{n-1}{r} v_r + \frac{1}{r^2} \operatorname{div}_\theta(\nabla_\theta v) \\ &= \partial_{rrrr} u + \frac{2(n-1)}{r} \partial_{rrr} u + \frac{(n-1)(n-3)}{r^2} \partial_{rr} u - \frac{(n-1)(n-3)}{r^3} \partial_r u \\ &\quad + r^{-4} \operatorname{div}_\theta(\nabla_\theta(\operatorname{div}_\theta(\nabla_\theta u))) \\ &\quad + 2r^{-2} \operatorname{div}_\theta(\nabla_\theta(u_{rr} + \frac{n-3}{r} u_r)) \\ &\quad - 2(n-4)r^{-4} \operatorname{div}_\theta(\nabla_\theta u).\end{aligned}$$

On ∂B_1 , we have

$$\begin{aligned}w &= \underbrace{\partial_{rrrr} u + 2(n-1)\partial_{rrr} u + (n-1)(n-3)\partial_{rr} u - (n-1)(n-3)\partial_r u}_{+ \operatorname{div}_\theta(\nabla_\theta(\operatorname{div}_\theta(\nabla_\theta u)))} \\ &\quad + \underbrace{2 \operatorname{div}_\theta(\nabla_\theta(u_{rr} + \frac{n-3}{r} u_r))}_{- 2(n-4) \operatorname{div}_\theta(\nabla_\theta u)} \\ &:= I(u) + J(u) + K(u) + L(u).\end{aligned}$$

By these notations, we can rewrite the term $\overline{E}_{d2}(u^\lambda, 1)$ appeared in (3.7) as following

$$\begin{aligned}
& \overline{E}_{d2}(u^\lambda, 1) \\
&= \int_{\partial B_1} (\lambda \frac{dw^\lambda}{d\lambda} \frac{du^\lambda}{d\lambda} - \lambda w^\lambda \frac{d^2 u^\lambda}{d\lambda^2}) - 5w^\lambda \frac{du^\lambda}{d\lambda} \\
&= \underbrace{\int_{\partial B_1} \lambda \frac{d}{d\lambda} I(u^\lambda) \frac{du^\lambda}{d\lambda} - \lambda I(u^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} - 5I(u^\lambda) \frac{du^\lambda}{d\lambda}}_{\text{I-term}} \\
&\quad + \underbrace{\int_{\partial B_1} \lambda \frac{d}{d\lambda} J(u^\lambda) \frac{du^\lambda}{d\lambda} - \lambda J(u^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} - 5J(u^\lambda) \frac{du^\lambda}{d\lambda}}_{\text{J-term}} \quad (3.22) \\
&\quad + \underbrace{\int_{\partial B_1} \lambda \frac{d}{d\lambda} K(u^\lambda) \frac{du^\lambda}{d\lambda} - \lambda K(u^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} - 5K(u^\lambda) \frac{du^\lambda}{d\lambda}}_{\text{K-term}} \\
&\quad + \underbrace{\int_{\partial B_1} \lambda \frac{d}{d\lambda} L(u^\lambda) \frac{du^\lambda}{d\lambda} - \lambda L(u^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} - 5L(u^\lambda) \frac{du^\lambda}{d\lambda}}_{\text{L-term}},
\end{aligned}$$

correspondingly, we rewrite $\overline{E}_{d2}(u^\lambda, 1)$ as

$$\begin{aligned}
\overline{E}_{d2}(u^\lambda, 1) &:= \mathcal{I} + \mathcal{J} + \mathcal{K} + \mathcal{L} \\
&:= I_1 + I_2 + I_3 + J_1 + J_2 + J_3 + K_1 + K_2 + K_3 + L_1 + L_2 + L_3,
\end{aligned}$$

where $I_1, I_2, I_3, J_1, J_2, J_3, K_1, K_2, K_3, L_1, L_2, L_3$ successively corresponding to the 12 terms in (3.22). By the conclusions of subsection 2.2, we have

$$\begin{aligned}
I(u^\lambda) &= \partial_{rrrr} u^\lambda + 2(n-1)\partial_{rrr} u^\lambda \\
&\quad + (n-1)(n-3)\partial_{rr} u^\lambda - (n-1)(n-3)\partial_r u^\lambda \\
&= \lambda^4 \frac{d^4 u^\lambda}{d\lambda^4} + \lambda^3 \left(2(n-1) - \frac{24}{p-1} \right) \frac{d^3 u^\lambda}{d\lambda^3} \\
&\quad + \lambda^2 \left[\frac{36}{p-1} \left(1 + \frac{6}{p-1} \right) - (n-1) \frac{36}{p-1} + (n-1)(n-3) \right] \frac{d^2 u^\lambda}{d\lambda^2} \\
&\quad + \lambda \left[-\frac{24}{p-1} \left(1 + \frac{6}{p-1} \right) \left(2 + \frac{6}{p-1} \right) + 2(n-1) \frac{18}{p-1} \left(1 + \frac{6}{p-1} \right) \right. \\
&\quad \left. + (n-1)(n-3) \left(-\frac{12}{p-1} - 1 \right) \right] \frac{du^\lambda}{d\lambda} \\
&\quad + \left[\left(1 + \frac{6}{p-1} \right) \left(2 + \frac{6}{p-1} \right) \left(3 + \frac{6}{p-1} \right) \frac{6}{p-1} \right. \\
&\quad \left. - (n-1) \left(1 + \frac{6}{p-1} \right) \left(2 + \frac{6}{p-1} \right) \frac{12}{p-1} \right. \\
&\quad \left. + (n-1)(n-3) \left(\frac{6}{p-1} + 2 \right) \frac{6}{p-1} \right] u^\lambda. \quad (3.23)
\end{aligned}$$

For convenience, we denote that

$$I(u^\lambda) = \lambda^4 \frac{d^4 u^\lambda}{d\lambda^4} + \lambda^3 \delta_1 \frac{d^3 u^\lambda}{d\lambda^3} + \lambda^2 \delta_2 \frac{d^2 u^\lambda}{d\lambda^2} + \lambda \delta_3 \frac{du^\lambda}{d\lambda} + \delta_4 u^\lambda, \quad (3.24)$$

where δ_i are the corresponding coefficients of $\lambda^i \frac{d^i u^\lambda}{d\lambda^i}$ appeared in (3.23) for $i = 1, 2, 3, 4$. Now taking the derivative of (3.24) with respect to λ we get

$$\begin{aligned} \frac{d}{d\lambda} I(u^\lambda) &= \lambda^4 \frac{d^5 u^\lambda}{d\lambda^5} + \lambda^3 (\delta_1 + 4) \frac{d^4 u^\lambda}{d\lambda^4} + \lambda^2 (3\delta_1 + \delta_2) \frac{d^3 u^\lambda}{d\lambda^3} \\ &\quad + \lambda (2\delta_2 + \delta_3) \frac{d^2 u^\lambda}{d\lambda^2} + (\delta_3 + \delta_4) \frac{du^\lambda}{d\lambda}. \end{aligned} \quad (3.25)$$

Since

$$\begin{aligned} &\partial_{rr} u^\lambda + (n-3) \partial_r u^\lambda \\ &= \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} + \lambda (n-3 - \frac{12}{p-1}) \frac{du^\lambda}{d\lambda} + \frac{6}{p-1} (4 + \frac{6}{p-1} - n) u^\lambda \\ &:= \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} + \lambda \alpha \frac{d^\lambda}{d\lambda} + \beta u^\lambda. \end{aligned} \quad (3.26)$$

Hence,

$$\frac{d}{d\lambda} [\partial_{rr} u^\lambda + (n-1) \partial_r u^\lambda] = \lambda^2 \frac{d^3 u^\lambda}{d\lambda^3} + \lambda (\alpha + 2) \frac{d^2 u^\lambda}{d\lambda^2} + (\alpha + \beta) \frac{du^\lambda}{d\lambda}, \quad (3.27)$$

here $\alpha = n - 3 - \frac{12}{p-1}$ and $\beta = \frac{6}{p-1} (4 + \frac{6}{p-1} - n)$.

3.4 The computations of I_1, I_2, I_3 and \mathcal{I}

We start with

$$\begin{aligned} I_1 &:= \int_{\partial B_1} \lambda \frac{d}{d\lambda} I(u^\lambda) \frac{du^\lambda}{d\lambda} \\ &= \int_{\partial B_1} \left(\lambda^5 \frac{d^5 u^\lambda}{d\lambda^5} + \lambda^4 (4 + \delta_1) \frac{d^4 u^\lambda}{d\lambda^4} + \lambda^3 (3\delta_1 + \delta_2) \frac{d^3 u^\lambda}{d\lambda^3} \right. \\ &\quad \left. + \lambda^2 (2\delta_2 + \delta_3) \frac{d^2 u^\lambda}{d\lambda^2} + \lambda (\delta_3 + \delta_4) \frac{du^\lambda}{d\lambda} \right) \frac{du^\lambda}{d\lambda} \\ &= \frac{d}{d\lambda} \int_{\partial B_1} \left[\lambda^5 \frac{d^4 u^\lambda}{d\lambda^4} \frac{du^\lambda}{d\lambda} - \lambda^5 \frac{d^3 u^\lambda}{d\lambda^3} \frac{d^2 u^\lambda}{d\lambda^2} + (\delta_1 - 1) \lambda^4 \frac{d^3 u^\lambda}{d\lambda^3} \frac{du^\lambda}{d\lambda} \right. \\ &\quad \left. + (4 - \delta_1 + \delta_2) \lambda^3 \frac{d^2 u^\lambda}{d\lambda^2} \frac{du^\lambda}{d\lambda} + \frac{3\delta_1 - \delta_2 + \delta_3 - 12}{2} \lambda^2 \left(\frac{du^\lambda}{d\lambda} \right)^2 \right] \\ &\quad + \int_{\partial B_1} \left[(12 - 3\delta_1 + \delta_2 + \delta_4) \lambda \left(\frac{du^\lambda}{d\lambda} \right)^2 \right. \\ &\quad \left. + (\delta_1 - 4 - \delta_2) \lambda^3 \left(\frac{d^2 u^\lambda}{d\lambda^2} \right)^2 + \lambda^5 \left(\frac{d^3 u^\lambda}{d\lambda^3} \right)^2 \right] \\ &\quad + \int_{\partial B_1} (6 - \delta_1) \lambda^4 \frac{d^3 u^\lambda}{d\lambda^3} \frac{d^2 u^\lambda}{d\lambda^2}, \end{aligned} \quad (3.28)$$

where $\delta_i, i = 1, 2, 3, 4$, are defined in (3.23) and (3.24). In this computation, we denote that $f = u^\lambda, f' := \frac{du^\lambda}{d\lambda}$ and we have used the fact that

$$\begin{aligned}\lambda^5 f'''' f' &= [\lambda^5 f'''' f' - \lambda^5 f''' f'' - 5\lambda^4 f''' f' + 20\lambda^3 f'' f' - 30\lambda^2 f' f']' \\ &\quad + 60\lambda(f')^2 - 20\lambda^3(f'')^2 + \lambda^5(f''')^2 + 10\lambda^4 f''' f'', \\ \lambda^4 f'''' f' &= [\lambda^4 f'''' f' - 4\lambda^3 f'' f' + 6\lambda^2 f' f']' - 12\lambda(f')^2 + 4\lambda^3(f'')^2 - \lambda^4 f''' f'', \\ \lambda^3 f''' f' &= [\lambda^3 f'' f' - \frac{3\lambda^2}{2} f' f']' + 3\lambda(f')^2 - \lambda^3(f'')^2,\end{aligned}$$

and

$$\lambda^2 f'' f' = [\frac{\lambda^2}{2} f' f']' - \lambda(f')^2.$$

$$\begin{aligned}I_2 &:= -\lambda \int_{\partial B_1} I(u_e^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} \\ &= -\lambda \int_{\partial B_1} (\lambda^4 \frac{d^4 u^\lambda}{d\lambda^4} + \lambda^3 \delta_1 \frac{d^3 u_e^\lambda}{d\lambda^3} + \lambda^2 \delta_2 \frac{d^2 u^\lambda}{d\lambda^2} + \lambda \delta_3 \frac{du^\lambda}{d\lambda} + \delta_4 u^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} \\ &= \frac{d}{d\lambda} \int_{\partial B_1} [-\lambda^5 \frac{d^3 u^\lambda}{d\lambda^3} \frac{d^2 u^\lambda}{d\lambda^2} - \delta_4 \lambda \frac{du^\lambda}{d\lambda} u^\lambda] \\ &\quad + \int_{\partial B_1} [\lambda^5 (\frac{d^3 u^\lambda}{d\lambda^3})^2 - \delta_2 \lambda^3 (\frac{d^2 u^\lambda}{d\lambda^2})^2 + \delta_4 \lambda (\frac{du^\lambda}{d\lambda})^2] \\ &\quad + \int_{\partial B_1} [(5 - \delta_1) \lambda^4 \frac{d^3 u^\lambda}{d\lambda^3} \frac{d^2 u^\lambda}{d\lambda^2} - \delta_3 \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} \frac{du^\lambda}{d\lambda} + \delta_4 \frac{du^\lambda}{d\lambda} u^\lambda],\end{aligned}\tag{3.29}$$

here we have used that

$$-\lambda^5 f'''' f'' = [-\lambda^5 f''' f'']' + 5\lambda^4 f''' f'' + \lambda^5(f''')^2$$

and

$$-\lambda f'' f = [-\lambda f' f]' + f' f + \lambda(f')^2.$$

Further,

$$\begin{aligned}I_3 &:= -5 \int_{\partial B_1} I(u^\lambda) \frac{du^\lambda}{d\lambda} \\ &= -5 \int_{\partial B_1} [\lambda^4 \frac{d^4 u^\lambda}{d\lambda^4} + \lambda^3 \delta_1 \frac{d^3 u^\lambda}{d\lambda^3} + \lambda^2 \delta_2 \frac{d^2 u^\lambda}{d\lambda^2} + \lambda \delta_3 \frac{du^\lambda}{d\lambda} + \delta_4 u^\lambda] \frac{du^\lambda}{d\lambda} \\ &= \frac{d}{d\lambda} \int_{\partial B_1} [-5 \frac{d^3 u^\lambda}{d\lambda^3} \frac{du^\lambda}{d\lambda} + (20 - 5\delta_1) \lambda^3 \frac{d^2 u^\lambda}{d\lambda^2} \frac{du^\lambda}{d\lambda}] \\ &\quad + \int_{\partial B_1} [(5\delta_1 - 20) \lambda^3 (\frac{d^2 u^\lambda}{d\lambda^2})^2 - 5\delta_3 \lambda (\frac{du^\lambda}{d\lambda})^2] \\ &\quad + \int_{\partial B_1} [5\lambda^4 \frac{d^3 u^\lambda}{d\lambda^3} \frac{d^2 u^\lambda}{d\lambda^2} + (15\delta_1 - 60 - 5\delta_2) \frac{d^2 u^\lambda}{d\lambda^2} \frac{du^\lambda}{d\lambda} - 5\delta_4 \frac{du^\lambda}{d\lambda} u^\lambda],\end{aligned}\tag{3.30}$$

here we have use that

$$-\lambda^4 f'''' f' = [-5\lambda^4 f''' f' + 20\lambda^3 f'' f']' - 20\lambda^3 (f'')^2 - 60\lambda^2 f'' f' + 5\lambda^4 f''' f''$$

and

$$-\lambda^3 f''' f' = [-\lambda^3 f'' f']' + 3\lambda^2 f'' f' + \lambda^3 (f'')^2.$$

Summing up I_1, I_2, I_3 , we can get the term \mathcal{I} .

$$\begin{aligned} \mathcal{I} &:= I_1 + I_2 + I_3 \\ &= \frac{d}{d\lambda} \int_{\partial B_1} \left[\lambda^5 \frac{d^4 u_e^\lambda}{d\lambda^4} \frac{du_e^\lambda}{d\lambda} - 2\lambda^5 \frac{d^3 u_e^\lambda}{d\lambda^3} \frac{d^2 u_e^\lambda}{d\lambda^2} \right. \\ &\quad + (\delta_1 - 6)\lambda^4 \frac{d^3 u^\lambda}{d\lambda^3} \frac{du^\lambda}{d\lambda} + (24 - 6\delta_1 + \delta_2)\lambda^3 \frac{d^2 u^\lambda}{d\lambda^2} \frac{du^\lambda}{d\lambda} \\ &\quad + \frac{3\delta_1 - \delta_2 + \delta_3 - 12}{2} \lambda^2 \frac{du^\lambda}{d\lambda} \frac{du^\lambda}{d\lambda} \\ &\quad + (\delta_1 - 8)\lambda^4 \left(\frac{d^2 u^\lambda}{d\lambda^2} \right)^2 - \delta_4 \lambda \frac{du^\lambda}{d\lambda} u^\lambda - 2\delta_4 (u^\lambda)^2 \Big] \\ &\quad + \int_{\partial B_1} \left[(10\delta_1 - 2\delta_2 - 56)\lambda^3 \left(\frac{d^2 u^\lambda}{d\lambda^2} \right)^2 \right. \\ &\quad + (12 - 3\delta_1 + \delta_2 - 5\delta_3 + 2\delta_4) \lambda \left(\frac{du^\lambda}{d\lambda} \right)^2 \Big] \\ &\quad + \int_{\partial B_1} \left[(15\delta_1 - 5\delta_2 - \delta_3 - 60)\lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} \frac{du^\lambda}{d\lambda} \right] \\ &\quad + 2\lambda^5 \int_{\partial B_1} \left(\frac{d^3 u^\lambda}{d\lambda^3} \right)^2. \end{aligned} \tag{3.31}$$

Since $u^\lambda(x) = \lambda^{\frac{6}{p-1}} u(\lambda x)$, we have the following

$$\begin{aligned} \lambda^4 \frac{d^4 u^\lambda}{d\lambda^4} &= \lambda^{\frac{6}{p-1}} \left[\frac{6}{p-1} \left(\frac{6}{p-1} - 1 \right) \left(\frac{6}{p-1} - 2 \right) \left(\frac{6}{p-1} - 3 \right) u(\lambda x) \right. \\ &\quad + \frac{24}{p-1} \left(\frac{6}{p-1} - 1 \right) \left(\frac{6}{p-1} - 2 \right) r \lambda \partial_r u(\lambda x) \\ &\quad + \frac{36}{p-1} \left(\frac{6}{p-1} - 1 \right) r^2 \lambda^2 \partial_{rr} u(\lambda x) \\ &\quad \left. + \frac{24}{p-1} r^3 \lambda^3 \partial_{rrr} u(\lambda x) + r^4 \lambda^4 \partial_{rrrr} u(\lambda x) \right], \end{aligned}$$

and

$$\begin{aligned} \lambda^3 \frac{d^3 u^\lambda}{d\lambda^3} &= \lambda^{\frac{6}{p-1}} \left[\frac{6}{p-1} \left(\frac{6}{p-1} - 1 \right) \left(\frac{6}{p-1} - 2 \right) u(\lambda x) \right. \\ &\quad + \frac{18}{p-1} \left(\frac{6}{p-1} - 1 \right) r \lambda \partial_r u(\lambda x) \\ &\quad \left. + \frac{18}{p-1} r^2 \lambda^2 \partial_{rr} u(\lambda x) + r^3 \lambda^3 \partial_{rrr} u(\lambda x) \right], \end{aligned}$$

$$\begin{aligned} & \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} \\ &= \lambda^{\frac{6}{p-1}} \left[\frac{6}{p-1} \left(\frac{6}{p-1} - 1 \right) u(\lambda x) + \frac{12}{p-1} r \lambda \partial_r u(\lambda x) + r^2 \lambda^2 \partial_{rr} u(\lambda x) \right] \end{aligned}$$

and

$$\lambda \frac{du^\lambda}{d\lambda} = \lambda^{\frac{6}{p-1}} \left[\frac{6}{p-1} u(\lambda x) + r \lambda \partial_r u(\lambda x) \right].$$

Hence, by scaling we have

$$\begin{aligned} & \frac{d}{d\lambda} \int_{\partial B_1} \lambda^5 \frac{d^4 u^\lambda}{d\lambda^4} \frac{du^\lambda}{d\lambda} \\ &= \frac{d}{d\lambda} \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} y^b \left[\frac{6}{p-1} \left(\frac{6}{p-1} - 1 \right) \left(\frac{6}{p-1} - 2 \right) \left(\frac{6}{p-1} - 3 \right) u \right. \\ &+ \frac{24}{p-1} \left(\frac{6}{p-1} - 1 \right) \left(\frac{6}{p-1} - 2 \right) \lambda \partial_r u + \frac{36}{p-1} \left(\frac{6}{p-1} - 1 \right) \lambda^2 \partial_{rr} u \\ &+ \left. \frac{24}{p-1} \lambda^3 \partial_{rrr} u + \lambda^4 \partial_{rrrr} u \right] \left[\frac{6}{p-1} u + r \lambda \partial_r u \right], \end{aligned}$$

further,

$$\begin{aligned} & \frac{d}{d\lambda} \int_{\partial B_1} \lambda^5 \frac{d^3 u^\lambda}{d\lambda^3} \frac{d^2 u^\lambda}{d\lambda^2} \\ &= \frac{d}{d\lambda} \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} \left[\frac{6}{p-1} \left(\frac{6}{p-1} - 1 \right) \left(\frac{6}{p-1} - 2 \right) u \right. \\ &+ \frac{18}{p-1} \left(\frac{6}{p-1} - 1 \right) \lambda \partial_r u + \frac{18}{p-1} \lambda^2 \partial_{rr} u + \lambda^3 \partial_{rrr} u \\ &+ \left. \left[\frac{6}{p-1} \left(\frac{6}{p-1} - 1 \right) u + \frac{12}{p-1} \lambda \partial_r u + \lambda^2 \partial_{rr} u \right] \right] \\ & \frac{d}{d\lambda} \int_{\partial B_1} \lambda^4 \frac{d^3 u^\lambda}{d\lambda^3} \frac{du^\lambda}{d\lambda} \\ &= \frac{d}{d\lambda} \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} \left[\frac{6}{p-1} \left(\frac{6}{p-1} - 1 \right) \left(\frac{6}{p-1} - 2 \right) u \right. \\ &+ \frac{18}{p-1} \left(\frac{6}{p-1} - 1 \right) \lambda \partial_r u + \frac{18}{p-1} \lambda^2 \partial_{rr} u + \lambda^3 \partial_{rrr} u \\ &+ \left. \left[\frac{6}{p-1} u + \lambda \partial_r u \right] \right]. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \frac{d}{d\lambda} \int_{\partial B_1} \lambda^3 \frac{d^2 u^\lambda}{d\lambda^2} \frac{du^\lambda}{d\lambda} \\ &= \frac{d}{d\lambda} \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} \left[\frac{6}{p-1} \left(\frac{6}{p-1} - 1 \right) u \right. \\ &+ \left. \frac{12}{p-1} \lambda \partial_r u + \lambda^2 \partial_{rr} u \right] \left[\frac{6}{p-1} u + \lambda \partial_r u \right], \end{aligned}$$

$$\frac{d}{d\lambda} \int_{\partial B_1} \lambda^2 \frac{du^\lambda}{d\lambda} \frac{du^\lambda}{d\lambda} = \frac{d}{d\lambda} \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} \left[\frac{6}{p-1} u + \lambda \partial_r u \right]^2,$$

$$\begin{aligned} \frac{d}{d\lambda} \int_{\partial B_1} \lambda^4 \frac{d^2 u^\lambda}{d\lambda^2} \frac{d^2 u^\lambda}{d\lambda^2} &= \frac{d}{d\lambda} \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} \left(\frac{6}{p-1} \left(\frac{6}{p-1} - 1 \right) u \right. \\ &\quad \left. + \frac{12}{p-1} \lambda \partial_r u + \lambda^2 \partial_{rr} u \right)^2, \end{aligned}$$

$$\frac{d}{d\lambda} \int_{\partial B_1} \lambda \frac{du^\lambda}{d\lambda} u^\lambda = \frac{d}{d\lambda} \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} \left[\frac{6}{p-1} u + \lambda \partial_r u \right] u,$$

and

$$\frac{d}{d\lambda} \int_{\partial B_1} u^\lambda = \frac{d}{d\lambda} \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} u^2.$$

3.5 The computations of $J_i, K_i, L_i (i = 1, 2, 3)$ and $\mathcal{J}, \mathcal{K}, \mathcal{L}$

We begin with

$$\begin{aligned} J_1 &:= \int_{\partial B_1} \lambda \frac{d}{d\lambda} J(u^\lambda) \frac{du^\lambda}{d\lambda} = \int_{\partial B_1} \lambda J \left(\frac{du^\lambda}{d\lambda} \right) \frac{du^\lambda}{d\lambda} \\ &= \lambda \int_{\partial B_1} \mathbf{div}_\theta \left(\nabla_\theta \left(\mathbf{div}_\theta \left(\nabla_\theta \frac{du^\lambda}{d\lambda} \right) \right) \right) \frac{du^\lambda}{d\lambda} \\ &= - \lambda \int_{\partial B_1} \nabla_\theta \left(\mathbf{div}_\theta \left(\nabla_\theta \frac{du^\lambda}{d\lambda} \right) \right) \nabla_\theta \frac{du^\lambda}{d\lambda} \\ &= \lambda \int_{\partial B_1} \left[\mathbf{div}_\theta \left(\nabla_\theta \frac{du^\lambda}{d\lambda} \right) \right]^2 \\ &= \lambda \int_{\partial B_1} \left[\frac{d}{d\lambda} \mathbf{div}_\theta \left(\nabla_\theta u^\lambda \right) \right]^2. \end{aligned} \tag{3.32}$$

Here we have used integrating by part formula on the unit sphere S^n . Next

$$\begin{aligned}
J_2 &:= -\lambda \int_{\partial B_1} J(u^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} \\
&= -\lambda \int_{\partial B_1} \mathbf{div}_\theta(\nabla_\theta(\mathbf{div}_\theta(\nabla_\theta u^\lambda))) \frac{d^2 u^\lambda}{d\lambda^2} \\
&= \lambda \int_{\partial B_1} \nabla_\theta(\mathbf{div}_\theta(\nabla_\theta u^\lambda)) \nabla_\theta \frac{d^2 u^\lambda}{d\lambda^2} \\
&= -\lambda \int_{\partial B_1} \mathbf{div}_\theta(\nabla_\theta u^\lambda) \frac{d^2}{d\lambda^2} \mathbf{div}_\theta(\nabla_\theta u^\lambda) \\
&= \frac{d}{d\lambda} \int_{\partial B_1} -\lambda [\mathbf{div}_\theta(\nabla_\theta u^\lambda)] \frac{d}{d\lambda} [\mathbf{div}_\theta(\nabla_\theta u^\lambda)] \\
&\quad + \int_{\partial B_1} \mathbf{div}_\theta(\nabla_\theta u^\lambda) \cdot \frac{d}{d\lambda} \mathbf{div}_\theta(\nabla_\theta u^\lambda) \\
&\quad + \lambda \int_{\partial B_1} \left[\frac{d}{d\lambda} \mathbf{div}_\theta(\nabla_\theta u^\lambda) \right]^2.
\end{aligned} \tag{3.33}$$

Here we denote that $g = \mathbf{div}_\theta(\nabla_\theta u^\lambda)$, $g' = \frac{d}{d\lambda} \mathbf{div}_\theta(\nabla_\theta u^\lambda)$ and we have used the fact that

$$-\lambda g g'' = [-\lambda g g']' + g g' + \lambda (g')^2 = [-\lambda g g' + \frac{1}{2} g^2]' + \lambda (g')^2.$$

Furthermore,

$$\begin{aligned}
J_3 &:= -5 \int_{\partial B_1} J(u^\lambda) \frac{du^\lambda}{d\lambda} \\
&= -5 \int_{\partial B_1} \mathbf{div}_\theta(\nabla_\theta(\mathbf{div}_\theta(\nabla_\theta u))) \frac{du^\lambda}{d\lambda} \\
&= 5 \int_{\partial B_1} \nabla_\theta(\mathbf{div}_\theta(\nabla_\theta u)) \nabla_\theta \frac{du^\lambda}{d\lambda} \\
&= -5 \int_{\partial B_1} \mathbf{div}_\theta(\nabla_\theta u) \frac{d}{d\lambda} \mathbf{div}_\theta(\nabla_\theta u).
\end{aligned} \tag{3.34}$$

Therefore, combining with (3.32), (3.33) and (3.34), we get that

$$\begin{aligned}
\mathcal{J} &:= J_1 + J_2 + J_3 \\
&= 2\lambda \int_{\partial B_1} \left[\frac{d}{d\lambda} \mathbf{div}_\theta(\nabla_\theta u^\lambda) \right]^2 \\
&\quad - 4 \int_{\partial B_1} \mathbf{div}_\theta(\nabla_\theta u) \frac{d}{d\lambda} \mathbf{div}_\theta(\nabla_\theta u) \\
&\quad + \frac{d}{d\lambda} \int_{\partial B_1} -\lambda [\mathbf{div}_\theta(\nabla_\theta u^\lambda)] \frac{d}{d\lambda} [\mathbf{div}_\theta(\nabla_\theta u^\lambda)] \\
&= 2\lambda \int_{\partial B_1} \left[\frac{d}{d\lambda} \mathbf{div}_\theta(\nabla_\theta u^\lambda) \right]^2 \\
&\quad - 2 \frac{d}{d\lambda} \int_{\partial B_1} \mathbf{div}_\theta(\nabla_\theta u^\lambda) \mathbf{div}_\theta(\nabla_\theta u^\lambda) \\
&\quad + \frac{d}{d\lambda} \int_{\partial B_1} -\lambda [\mathbf{div}_\theta(\nabla_\theta u^\lambda)] \frac{d}{d\lambda} [\mathbf{div}_\theta(\nabla_\theta u^\lambda)].
\end{aligned} \tag{3.35}$$

Hence, we get that

$$\begin{aligned}
\mathcal{J} &\geq -2 \frac{d}{d\lambda} \int_{\partial B_1} \mathbf{div}_\theta(\nabla_\theta u) \mathbf{div}_\theta(\nabla_\theta u) \\
&\quad + \frac{d}{d\lambda} \int_{\partial B_1} -\lambda [\mathbf{div}_\theta(\nabla_\theta u^\lambda)] \frac{d}{d\lambda} [\mathbf{div}_\theta(\nabla_\theta u)] \\
&= -2 \frac{d}{d\lambda} \int_{\partial B_1} [\mathbf{div}_\theta(\nabla_\theta u)]^2 \\
&\quad + \frac{d}{d\lambda} \int_{\partial B_1} -\lambda \frac{d}{d\lambda} [\mathbf{div}_\theta(\nabla_\theta u)]^2.
\end{aligned} \tag{3.36}$$

Note that

$$\begin{aligned}
&\frac{d}{d\lambda} \int_{\partial B_1} [\mathbf{div}_\theta(\nabla_\theta u^\lambda)]^2 \\
&= \frac{d}{d\lambda} \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} (\lambda^2 \Delta u - \lambda^2 \partial_{rr} u - (n-1)\lambda \partial_r u)^2,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{d\lambda} \int_{\partial B_1} \lambda \frac{d}{d\lambda} [\mathbf{div}_\theta(\nabla_\theta u^\lambda)]^2 \\
&= \frac{d}{d\lambda} \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-4} \frac{d}{d\lambda} (\lambda^2 \Delta u - \lambda^2 \partial_{rr} u - (n-1)\lambda \partial_r u)^2.
\end{aligned}$$

Next we compute K_1, K_2, K_3 and \mathcal{K} .

$$\begin{aligned}
K_1 &:= \lambda \int_{\partial B_1} \frac{d}{d\lambda} K(u^\lambda) \frac{du^\lambda}{d\lambda} \\
&= 2\lambda \int_{\partial B_1} \mathbf{div}_\theta \left(\nabla_\theta \left(\frac{d}{d\lambda} (u_{rr} + (n-1)u_r) \right) \right) \frac{du^\lambda}{d\lambda} \\
&= 2\lambda \int_{\partial B_1} \mathbf{div}_\theta \left(\nabla_\theta \left(\lambda^3 \frac{d^3 u^\lambda}{d\lambda^3} + \lambda^2 (\alpha + 2) \frac{d^2 u^\lambda}{d\lambda^2} + \lambda (\alpha + \beta) \frac{du^\lambda}{d\lambda} \right) \right) \frac{du^\lambda}{d\lambda} \\
&= -2\lambda \int_{\partial B_1} \nabla_\theta \left(\lambda^3 \frac{d^3 u^\lambda}{d\lambda^3} + \lambda^2 (\alpha + 2) \frac{d^2 u^\lambda}{d\lambda^2} + \lambda (\alpha + \beta) \frac{du^\lambda}{d\lambda} \right) \nabla_\theta \frac{du^\lambda}{d\lambda} \\
&= \frac{d}{d\lambda} \left(\int_{\partial B_1} -\lambda^3 \frac{d}{d\lambda} \left(\frac{d}{d\lambda} \nabla_\theta u^\lambda \right)^2 \right) + (2 - 2\alpha) \lambda^2 \int_{\partial B_1} \left(\frac{d}{d\lambda} \nabla_\theta u^\lambda \right) \\
&\quad \cdot \left(\frac{d^2}{d\lambda^2} \nabla_\theta u^\lambda \right) + 2\lambda^3 \int_{\partial B_1} \left(\frac{d^2}{d\lambda^2} \nabla_\theta u^\lambda \right)^2 \\
&\quad - (2\alpha + 2\beta) \lambda \int_{\partial B_1} \left(\frac{d}{d\lambda} \nabla_\theta u^\lambda \right)^2.
\end{aligned} \tag{3.37}$$

Here we denote that $h = \nabla_\theta u^\lambda, h' = \frac{d}{d\lambda} \nabla_\theta u^\lambda$, and used that

$$-\lambda^3 h' h''' = \left[-\frac{\lambda^3}{2} \frac{d}{d\lambda} (h')^2 \right]' + 3\lambda^2 h' h'' + \lambda^3 (h'')^2.$$

Next,

$$\begin{aligned}
K_2 &:= -\lambda \int_{\partial B_1} K(u^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} \\
&= -2\lambda \int_{\partial B_1} \mathbf{div}_\theta \left(\nabla_\theta \left(\lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} + \lambda \alpha \frac{du^\lambda}{d\lambda} + \beta u^\lambda \right) \right) \frac{d^2 u^\lambda}{d\lambda^2} \\
&= 2\lambda \int_{\partial B_1} \nabla_\theta \left(\lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} + \lambda \alpha \frac{du^\lambda}{d\lambda} + \beta u^\lambda \right) \nabla_\theta \frac{d^2 u^\lambda}{d\lambda^2} \\
&= \frac{d}{d\lambda} \int_{\partial B_1} \left[2\beta \lambda \nabla_\theta u^\lambda \frac{d}{d\lambda} \nabla_\theta u^\lambda - \beta (\nabla_\theta u^\lambda)^2 \right] \\
&\quad + 2\lambda^3 \int_{\partial B_1} \left(\frac{d^2}{d\lambda^2} \nabla_\theta u^\lambda \right)^2 - 2\lambda \beta \int_{\partial B_1} \left(\frac{d}{d\lambda} \nabla_\theta u^\lambda \right)^2 \\
&\quad + 2\lambda^2 \alpha \int_{\partial B_1} \frac{d}{d\lambda} \nabla_\theta u^\lambda \frac{d^2}{d\lambda^2} \nabla_\theta u^\lambda.
\end{aligned} \tag{3.38}$$

Here we have used that

$$2\lambda h h'' = [2\lambda h h' - h^2]' - 2\lambda (h')^2.$$

Further,

$$\begin{aligned}
K_3 &:= -5 \int_{\partial B_1} K(u^\lambda) \frac{du^\lambda}{d\lambda} \\
&= -10 \int_{\partial B_1} \mathbf{div}_\theta \left(\nabla_\theta \left(\lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} + \lambda \alpha \frac{du^\lambda}{d\lambda} + \beta u^\lambda \right) \right) \frac{du^\lambda}{d\lambda} \\
&= 10 \int_{\partial B_1} \nabla_\theta \left(\lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} + \lambda \alpha \frac{du^\lambda}{d\lambda} + \beta u^\lambda \right) \nabla_\theta \frac{du^\lambda}{d\lambda} \\
&= \frac{d}{d\lambda} \left[5\beta \int_{\partial B_1} \nabla_\theta u^\lambda \nabla_\theta u^\lambda \right] + 10\lambda\alpha \int_{\partial B_1} \left(\frac{d}{d\lambda} \nabla_\theta u^\lambda \right)^2 \\
&\quad + 10\lambda^2 \int_{\partial B_1} \frac{d}{d\lambda} \nabla_\theta u^\lambda \frac{d^2}{d\lambda^2} \nabla_\theta u^\lambda.
\end{aligned} \tag{3.39}$$

Now combine with (3.37), (3.38) and (3.39), we get that

$$\begin{aligned}
\mathcal{K} &:= K_1 + K_2 + K_3 \\
&= \frac{d}{d\lambda} \int_{\partial B_1} \left[-\lambda^3 \frac{d}{d\lambda} \left(\frac{d}{d\lambda} \nabla_\theta u^\lambda \right)^2 \right. \\
&\quad \left. + 2\beta\lambda \nabla_\theta u^\lambda \frac{d}{d\lambda} \nabla_\theta u^\lambda + 4\beta (\nabla_\theta u^\lambda)^2 \right] \\
&\quad + 4\lambda^3 \int_{\partial B_1} \left(\frac{d^2}{d\lambda^2} \nabla_\theta u^\lambda \right)^2 + (8\alpha - 4\beta)\lambda \int_{\partial B_1} \left(\frac{d}{d\lambda} \nabla_\theta u^\lambda \right)^2 \\
&\quad + 12\lambda^2 \int_{\partial B_1} \frac{d}{d\lambda} \nabla_\theta u^\lambda \frac{d^2}{d\lambda^2} \nabla_\theta u^\lambda. \\
&= \frac{d}{d\lambda} \int_{\partial B_1} \left[-\lambda^3 \frac{d}{d\lambda} \left(\frac{d}{d\lambda} \nabla_\theta u^\lambda \right)^2 + 6\lambda^2 \left(\nabla_\theta \frac{du^\lambda}{d\lambda} \right)^2 \right. \\
&\quad \left. + 2\beta\lambda \nabla_\theta u^\lambda \frac{d}{d\lambda} \nabla_\theta u^\lambda + 4\beta (\nabla_\theta u^\lambda)^2 \right] \\
&\quad + \int_{\partial B_1} 4\lambda^3 \left(\frac{d^2}{d\lambda^2} \nabla_\theta u^\lambda \right)^2 + (8\alpha - 4\beta - 12) \left(\frac{d}{d\lambda} \nabla_\theta u^\lambda \right)^2
\end{aligned} \tag{3.40}$$

Notice that by scaling we have

$$\begin{aligned}
\frac{d}{d\lambda} \int_{\partial B_1} (\nabla_\theta u^\lambda)^2 &= \frac{d}{d\lambda} \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} [\lambda^2 |\nabla u|^2 - \lambda^2 |\partial_r u|^2], \\
\frac{d}{d\lambda} \int_{\partial B_1} \lambda \frac{d}{d\lambda} (\nabla_\theta u^\lambda)^2 &= \frac{d}{d\lambda} \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-4} \frac{d}{d\lambda} [\lambda^2 |\nabla u|^2 - \lambda^2 |\partial_r u|^2]
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{d\lambda} \int_{\partial B_1} \lambda^3 \frac{d}{d\lambda} \left(\frac{d}{d\lambda} \nabla_\theta u^\lambda \right)^2 &= \frac{d}{d\lambda} \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-2} \frac{d}{d\lambda} \left[\nabla \left(\frac{6}{p-1} u + \lambda \partial_r u \right)^2 \right. \\
&\quad \left. - |\partial_r \left(\frac{6}{p-1} u + \lambda \partial_r u \right)|^2 \right]^2.
\end{aligned}$$

Finally, we compute \mathcal{L} .

$$\begin{aligned} L_1 &:= \int_{\partial B_1} \lambda \frac{d}{d\lambda} L(u^\lambda) \frac{du^\lambda}{d\lambda} = -2(n-4)\lambda \int_{\partial B_1} \mathbf{div}_\theta(\nabla_\theta \frac{du^\lambda}{d\lambda}) \frac{du^\lambda}{d\lambda} \\ &= 2(n-4)\lambda \int_{\partial B_1} (\nabla_\theta \frac{du^\lambda}{d\lambda})^2; \end{aligned}$$

$$\begin{aligned} L_2 &:= \int_{\partial B_1} -\lambda L(u^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} = 2(n-4)\lambda \int_{\partial B_1} \mathbf{div}_\theta(\nabla_\theta u^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} \\ &= -2(n-4) \int_{\partial B_1} \lambda \nabla_\theta u^\lambda \frac{d^2}{d\lambda^2} \nabla_\theta u^\lambda \\ &= (4-n) \int_{\partial B_1} \frac{d}{d\lambda} [2\lambda \nabla_\theta u^\lambda \nabla_\theta \frac{du^\lambda}{d\lambda} - (\nabla_\theta u^\lambda)^2] + 2(n-4)\lambda \int_{\partial B_1} [\frac{d}{d\lambda} \nabla_\theta u^\lambda]^2; \end{aligned}$$

$$\begin{aligned} L_3 &:= \int_{\partial B_1} -5L(u^\lambda) \frac{du^\lambda}{d\lambda} = 10(n-4) \int_{\partial B_1} \mathbf{div}_\theta(\nabla_\theta u^\lambda) \frac{du^\lambda}{d\lambda} \\ &= -10(n-4) \int_{\partial B_1} \nabla_\theta u^\lambda \nabla_\theta \frac{du^\lambda}{d\lambda} = -5(n-4) \frac{d}{d\lambda} \int_{\partial B_1} [\nabla_\theta u^\lambda]^2. \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{L} &:= L_1 + L_2 + L_3 \\ &= -(n-4) \frac{d}{d\lambda} \int_{\partial B_1} [\lambda \frac{d}{d\lambda} (\nabla_\theta u^\lambda)^2] - 4(n-4) \frac{d}{d\lambda} \int_{\partial B_1} [\nabla_\theta u^\lambda]^2 \\ &\quad + 4(n-4)\lambda \int_{\partial B_1} [\frac{d}{d\lambda} \nabla_\theta u^\lambda]^2 \\ &= -(n-4) \frac{d}{d\lambda} \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-4} \frac{d}{d\lambda} [\lambda^2 |\nabla u|^2 - \lambda^2 |\partial_r u|^2] \\ &\quad - 4(n-4) \frac{d}{d\lambda} \int_{\partial B_\lambda} \lambda^{6\frac{p+1}{p-1}-n-5} [\lambda^2 |\nabla u|^2 - \lambda^2 |\partial_r u|^2] \\ &\quad + 4(n-4)\lambda \int_{\partial B_1} [\frac{d}{d\lambda} \nabla_\theta u^\lambda]^2. \end{aligned}$$

Proof of Theorem 2.1. From the equation (3.7) and combining with the estimates on $\mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}$ and the basic integrate by part, we obtain Theorem 2.1. \square

4 Homogeneous solutions

We first deduce polar coordinate representation for the harmonic, biharmonic and triharmonic operator:

$$\Delta u = (\partial_{rr} + \frac{n-1}{r})u + \frac{1}{r^2}\Delta_\theta u,$$

and

$$\begin{aligned}\Delta^2 u &= (\partial_{rr} + \frac{n-1}{r}\partial_r)^2 u + \Delta_\theta (\partial_{rr} + \frac{n-1}{r}\partial_r)(r^{-2}u) \\ &\quad + \Delta_\theta (r^{-2}(\partial_{rr} + \frac{n-1}{r}\partial_r)u) + \Delta_\theta^2 (r^{-4}u).\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta^3 u &= (\partial_{rr} + \frac{n-1}{r}\partial_r)^3 u + \Delta_\theta^3 (r^{-6}u) \\ &\quad + \Delta_\theta \left((\partial_{rr} + \frac{n-1}{r}\partial_r)(r^{-2}(\partial_{rr} + \frac{n-1}{r}\partial_r)u) \right. \\ &\quad \left. + (\partial_{rr} + \frac{n-1}{r}\partial_r)^2 (r^{-2}u) + r^{-2}(\partial_{rr} + \frac{n-1}{r}\partial_r)^2 u \right) \\ &\quad + \Delta_\theta^2 \left((\partial_{rr} + \frac{n-1}{r}\partial_r)(r^{-4}u) + r^{-4}(\partial_{rr} + \frac{n-1}{r}\partial_r)u \right. \\ &\quad \left. + r^{-2}(\partial_{rr} + \frac{n-1}{r}\partial_r)(r^{-2}u) \right).\end{aligned}$$

Assume that u is homogeneous, that is, $u = r^{-\frac{6}{p-1}}w(\theta)$. By a direct calculation, the function w satisfy

$$\Delta_\theta^3 w - k_2 \Delta_\theta^2 w + k_1 \Delta_\theta w - k_0 w = |w|^{p-1}w, \quad (4.1)$$

where

$$\begin{aligned}k_0 &= \frac{6}{p-1}(\frac{6}{p-1} + 2)(\frac{6}{p-1} + 4)(n-2 - \frac{6}{p-1})(n-4 - \frac{6}{p-1})(n-6 - \frac{6}{p-1}), \\ k_1 &= \frac{6}{p-1}(\frac{6}{p-1} + 2 - n)(\frac{6}{p-1} + 4)(\frac{6}{p-1} + 6 - n) \\ &\quad + (\frac{6}{p-1} + 2)(\frac{6}{p-1} + 4 - n)(\frac{6}{p-1} + 4)(\frac{6}{p-1} + 6 - n) \\ &\quad + \frac{6}{p-1}(\frac{6}{p-1} + 2 - n)(\frac{6}{p-1} + 2)(\frac{6}{p-1} + 4 - n). \\ k_2 &= (\frac{6}{p-1} + 4)(n - \frac{6}{p-1} - 6) + \frac{6}{p-1}(n - \frac{6}{p-1} - 2) \\ &\quad + (k+2)(n - \frac{6}{p-1} - 4).\end{aligned}$$

Hence, test (4.1) with $w(\theta)$, we have

$$\int_{S^{n-1}} |\nabla_\theta \Delta_\theta w|^2 + k_2 |\Delta_\theta w|^2 + k_1 |\nabla_\theta w|^2 + k_0 w^2 = \int_{S^{n-1}} |w|^{p+1}. \quad (4.2)$$

For any $\varepsilon > 0$, choose any $\eta_\varepsilon \in C_0^\infty(\frac{\varepsilon}{2}, \frac{2}{\varepsilon})$, such that $\eta_\varepsilon = 1$ in $(\varepsilon, \frac{1}{\varepsilon})$ and

$$r|\eta'_\varepsilon(r)| + r^2|\eta''_\varepsilon(r)| + r^3|\eta'''_\varepsilon(r)| + r^4|\eta''''_\varepsilon(r)| + r^5|\eta'''''_\varepsilon(r)| + r^6|\eta''''''_\varepsilon(r)| \leq 1000.$$

Note that

$$\Delta(r^{-\frac{n-6}{2}}w(\theta)\eta_\varepsilon(r)) = (\partial_{rr} + \frac{n-1}{r}\partial_r)(r^{-\frac{n-6}{2}}\eta_\varepsilon(r))w(\theta) + r^{-\frac{n-2}{2}}\eta_\varepsilon(r)w(\theta)$$

and that $|\nabla u|^2 = \frac{1}{r^2}|\nabla_\theta u|^2 + u_r^2$, we see that

$$\begin{aligned} & |\nabla \Delta(r^{-\frac{n-6}{2}}w(\theta)\eta_\varepsilon(r))|^2 \\ &= \frac{1}{r^2}|\nabla_\theta[(\partial_{rr} + \frac{n-1}{r}\partial_r)(r^{-\frac{n-6}{2}}\eta_\varepsilon(r))w(\theta) + r^{-\frac{n-2}{2}}\eta_\varepsilon(r)\Delta_\theta w]|^2 \\ &\quad + |\frac{\partial}{\partial r}[(\partial_{rr} + \frac{n-1}{r}\partial_r)(r^{-\frac{n-6}{2}}\eta_\varepsilon(r))w(\theta) + r^{-\frac{n-2}{2}}\eta_\varepsilon(r)\Delta_\theta w]|^2 \\ &:= I_1 + I_2. \end{aligned}$$

By a straightforward calculation we have the following estimates

$$\begin{aligned} \int_{\mathbb{R}^n} I_1 &\leq \left(\int_{S^{n-1}} |\nabla_\theta \Delta_\theta w|^2 + \frac{(n-6)(n+2)}{2} |\Delta_\theta|^2 + \frac{(n-6)^2(n+2)^2}{16} |\nabla_\theta w|^2 d\theta \right) \\ &\quad \cdot \left(\int_0^\infty r^{-1} \eta_\varepsilon^2(r) dr \right) \\ &\quad + C \left(\int_{S^{n-1}} |\nabla_\theta w|^2 + |\Delta_\theta w|^2 d\theta \right) \\ &\quad \cdot \left(\int_0^\infty \sum_{1 \leq k+j \leq 4, k, j \geq 0} r^{k+j-1} \eta_\varepsilon^{(k)} \eta_\varepsilon^{(j)} dr \right) \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \int_{\mathbb{R}^n} I_2 &\leq \left(\int_{S^{n-1}} \frac{(n-2)^2}{4} |\Delta_\theta w|^2 + \frac{(n-6)(n-2)^2(n+2)}{8} |\nabla_\theta w|^2 \right. \\ &\quad \left. + \frac{(n-6)^2(n-2)^2(n+2)^2}{64} w^2 d\theta \right) \cdot \left(\int_0^\infty r^{-1} \eta_\varepsilon^2(r) dr \right) \\ &\quad + C \left(\int_{S^{n-1}} w^2(\theta) + |\nabla_\theta w|^2 \right) \cdot \left(\int_0^\infty \sum_{1 \leq k+j \leq 6, k, j \geq 0} r^{k+j-1} \eta_\varepsilon^{(k)} \eta_\varepsilon^{(j)} dr \right). \end{aligned} \tag{4.4}$$

Here $\eta_\varepsilon^{(k)} := \frac{d^k}{dr^k} \eta_\varepsilon$ for $k \geq 1$ and $\eta_\varepsilon^{(0)} := \eta_\varepsilon$. Notice that

$$\begin{aligned} \int_0^\infty r^{-1} \eta_\varepsilon^2(r) dr &\geq |\log \varepsilon| \rightarrow +\infty, \text{ as } \varepsilon \rightarrow 0^+, \\ \int_0^\infty \sum_{1 \leq k+j \leq 6, k, j \geq 0} r^{k+j-1} \eta_\varepsilon^{(k)} \eta_\varepsilon^{(j)} dr &\leq C, \end{aligned}$$

where C is independent of the radius R . Recall the stability condition for triharmonic equation:

$$\int_{\mathbb{R}^n} |\nabla \Delta \phi|^2 dx \geq p \int_{\mathbb{R}^n} |u|^{p-1} \phi^2 dx.$$

Let $\phi = r^{-\frac{n-6}{2}} w(\theta) \eta_\varepsilon(r)$. Then combining with (4.3) and (4.4) and letting $\varepsilon \rightarrow 0$, we obtain that

$$\begin{aligned} p \int_{s^{n-1}} |w|^{p+1} d\theta &\leq \int_{s^{n-1}} |\nabla_\theta \Delta_\theta w|^2 + \frac{3n^2 - 12n - 20}{4} |\Delta_\theta w|^2 \\ &\quad + \frac{(n-6)(n+2)(3n^2 - 12n - 4)}{16} |\nabla_\theta w|^2 + \frac{(n-6)^2(n-2)^2(n+2)^2}{64} w^2(\theta), \end{aligned}$$

this combine with (4.2), we have the following estimate

$$\begin{aligned} &\int_{s^{n-1}} (p-1) |\nabla_\theta \Delta_\theta w|^2 + (pk_2 - \frac{3n^2 - 12n - 20}{4}) |\Delta_\theta w|^2 \\ &\quad + (pk_1 - \frac{(n-6)(n+2)(3n^2 - 12n - 4)}{16}) |\nabla_\theta w|^2 \\ &\quad + (pk_0 - \frac{(n-6)^2(n-2)^2(n+2)^2}{64}) w^2(\theta) \leq 0. \end{aligned} \tag{4.5}$$

Since, $k = \frac{6}{p-1}$, we have $p = \frac{k+6}{k}$. Equivalently, by the coefficient of the above inequality, we let

$$\begin{aligned} c_0 &:= (k+6)k_0/k - \frac{(n-6)^2(n-2)^2(n+2)^2}{64}, \\ c_1 &:= (k+6)k_1 - \frac{(n-6)(n+2)(3n^2 - 12n - 4)}{16} k, \\ c_2 &:= (k+6)k_2 - \frac{3n^2 - 12n - 20}{4} k. \end{aligned} \tag{4.6}$$

We consider the algebraic equation c_0, c_1, c_2 about the variable k , we only need consider positive real roots. Roughly speaking, c_0 is a six-order algebraic equation about k , it has no explicit solution in general. Nonetheless we shall prove

Lemma 4.1. *Assume that $\frac{n+6}{n-6} < p < p_c(n)$. Then $c_0, c_1, c_2 > 0$.*

Assuming the validity of Lemma 4.1, we derive from (4.5) that $w \equiv 0$. This gives

Theorem 4.1. *Let $u \in W_{loc}^{3,2}(\mathbb{R}^n \setminus \{0\})$ be a homogeneous, stable solution of (1.1), for $\frac{n+6}{n-6} < p < p_c(n)$. Assume that $|u|^{p+1} \in L_{loc}^1(\mathbb{R}^n \setminus \{0\})$, then $u \equiv 0$.*

The proof of Lemma 4.1 is quite technical and thus we delay it to the appendix.

5 Energy estimates and Blow down analysis

In the first part of this section, we obtain initial energy estimates on the solutions of (1.1), which are important when we perform a blow-down analysis in the second part of this section.

5.1 Energy estimates

Lemma 5.1. *Let u be a stable solution of (1.1), then there exists a positive constant C such that*

$$\begin{aligned}
& \int_{\mathbb{R}^n} |u|^{p+1} \eta^6 + \int_{\mathbb{R}^n} |\nabla \Delta u|^2 \eta^6 \\
& \leq C \left(\int_{\mathbb{R}^n} |\Delta u|^2 \eta^4 |\nabla \eta|^2 + \int_{\mathbb{R}^n} |\nabla u|^2 \frac{|\Delta \eta^6|^2}{\eta^6} + \int_{\mathbb{R}^n} u^2 \frac{|\nabla \Delta \eta^6|^2}{\eta^6} \right. \\
& \quad + \int_{\mathbb{R}^n} |\nabla u|^2 \eta^2 |\nabla \eta|^4 + \int_{\mathbb{R}^n} |\nabla^2 u|^2 \eta^4 |\nabla \eta|^2 \\
& \quad \left. + \int_{\mathbb{R}^n} |\nabla u|^2 \eta^4 |\nabla^2 \eta|^2 \right). \tag{5.1}
\end{aligned}$$

Proof. Multiplying the equation (1.1) with $u\eta^6$, where η is a test function, we get that

$$\begin{aligned}
\int_{\mathbb{R}^n} |u|^{p+1} \eta^6 &= \int_{\mathbb{R}^n} -\Delta^3 u \cdot u \eta^6 = \int_{\mathbb{R}^n} \nabla \Delta^2 u \cdot \nabla (u \eta^6) = - \int_{\mathbb{R}^n} \Delta^2 u \Delta (u \eta^6) \\
&= \int_{\mathbb{R}^n} \nabla \Delta u \cdot \nabla \Delta (u \eta^6). \tag{5.2}
\end{aligned}$$

Since $\Delta(\xi\eta) = \eta\Delta\xi + \xi\Delta\eta + 2\nabla\xi\nabla\eta$, we have

$$\Delta(u\eta^6) = \eta^6 \Delta u + u \Delta \eta^6 + 12\eta^5 \nabla u \nabla \eta,$$

therefore,

$$\begin{aligned}
\nabla \Delta (u\eta^6) \nabla \Delta u &= 6\eta^5 \Delta u \nabla \eta \nabla \Delta u + (\eta)^6 (\nabla \Delta u)^2 + \Delta \eta^6 \nabla u \nabla \Delta u \\
&\quad + u \nabla \Delta \eta^6 \nabla \Delta u + 60\eta^4 (\nabla \eta \nabla \Delta u) (\nabla u \nabla \eta) \\
&\quad + 12\eta^5 \sum_{i,j} \partial_{ij} u \partial_i \eta \partial_j \Delta u + 12\eta^5 \sum_{i,j} \partial_i u \partial_{ij} \eta \partial_j \Delta u, \tag{5.3}
\end{aligned}$$

where $\partial_j (j = 1, \dots, n)$ denote the derivatives with respect to x_1, \dots, x_n respectively. A similar way can be applied to deal with the following term $|\nabla \Delta (u\eta^3)|^2$. On the other hand, by the stability condition (see Definition 1.1), we have

$$p \int_{\mathbb{R}^n} |u|^{p+1} \eta^6 \leq \int_{\mathbb{R}^n} |\nabla \Delta (u\eta^3)|^2. \tag{5.4}$$

Combine this with (5.2), (5.3) and (5.4), we have

$$\begin{aligned}
& \int_{\mathbb{R}^n} |\nabla \Delta u|^2 \eta^6 \\
& \leq C\varepsilon \int_{\mathbb{R}^n} (\nabla \Delta u)^2 \eta^6 + C(\varepsilon) \left[\int_{\mathbb{R}^n} |\Delta u|^2 \eta^4 |\nabla \eta|^2 \right. \\
& \quad + \int_{\mathbb{R}^n} |\nabla u|^2 \left(\frac{|\Delta \eta^6|^2}{\eta^6} + \eta^4 |\nabla^2 \eta|^2 \right) \\
& \quad \left. + \int_{\mathbb{R}^n} u^2 \frac{|\nabla \Delta \eta^6|^2}{\eta^6} + \int_{\mathbb{R}^n} |\nabla u|^2 \eta^2 |\nabla \eta|^4 + \int_{\mathbb{R}^n} |\nabla^2 u|^2 \eta^4 |\nabla \eta|^2 \right],
\end{aligned}$$

we can select ε so small that $C\varepsilon \leq \frac{1}{2}$. Finally, combine with (5.2) and (5.3), we obtain the conclusion of this lemma. \square

Lemma 5.2. *Let u be a stable solution of (1.1). Then*

$$\int_{B_R} |u|^{p+1} + \int_{B_R} (\nabla \Delta u)^2 \leq CR^{-6} \int_{B_{2R}} u^2, \quad (5.5)$$

$$\int_{B_R} |u|^{p+1} + \int_{B_R} (\nabla \Delta u)^2 \leq CR^{n-6\frac{p+1}{p-1}}. \quad (5.6)$$

Proof. We let $\eta = \xi^m$ where $m > 1$ in the estimate (5.1), we have

$$\begin{aligned} & \int_{\mathbb{R}^n} |\nabla \Delta u|^2 \xi^{6m} + \int_{\mathbb{R}^n} |u|^{p+1} \xi^{6m} \\ & \leq \int_{\mathbb{R}^n} u^2 g_0(\xi) + \int_{\mathbb{R}^n} |\nabla u|^2 g_1(\xi) + \int_{\mathbb{R}^n} |\Delta u|^2 g_2(\xi), \end{aligned} \quad (5.7)$$

where

$$\begin{aligned} g_0(\xi) &:= \xi^{6m-6} \sum_{0 \leq i+j+k+r+s+t \leq 6} |\nabla^i \xi| |\nabla^j \xi| |\nabla^k \xi| |\nabla^r \xi| |\nabla^s \xi| |\nabla^t \xi|, \\ g_1(\xi) &:= \xi^{6m-4} \sum_{0 \leq i+j+k+l \leq 4} |\nabla^i \xi| |\nabla^j \xi| |\nabla^k \xi| |\nabla^l \xi|, \\ g_2(\xi) &:= \xi^{6m-2} \sum_{0 \leq i+j \leq 2} |\nabla^i \xi| |\nabla^j \xi|, \end{aligned} \quad (5.8)$$

where we define $\nabla^0 \xi := \xi$ and notice that $g_m(\xi) \geq 0$ for $m = 0, 1, 2$. Now, we claim that

$$g_1^2(\xi) \leq C g_0(\xi) g_2(\xi), \quad |\nabla^2 g_2(\xi)| \leq C g_1(\xi), \quad g_2^2(\xi) \leq C \xi^{6m} g_1(\xi). \quad (5.9)$$

This claim can be verified by direct calculations and will be used for the following estimates. Since $|\nabla u|^2 = \frac{1}{2} \Delta(u^2) - u \Delta u$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla u|^2 g_1(\xi) &= \frac{1}{2} \int_{\mathbb{R}^n} \Delta(u^2) g_1(\xi) - \int_{\mathbb{R}^n} u \Delta u g_1(\xi) \\ &= \frac{1}{2} \int_{\mathbb{R}^n} u^2 \Delta g_1(\xi) - \int_{\mathbb{R}^n} u \Delta u g_1(\xi) \\ &\leq \frac{1}{2} \int_{\mathbb{R}^n} u^2 \Delta g_1(\xi) + \varepsilon \int_{\mathbb{R}^n} (\Delta u)^2 g_2(\xi) + \frac{1}{4\varepsilon} \int_{\mathbb{R}^n} u^2 g_0(\xi). \end{aligned} \quad (5.10)$$

We note the following differential identity

$$(\Delta u)^2 = \sum_{j,k} (u_j u_k)_{jk} - \sum_{j,k} (u_{jk})^2 - 2 \nabla \Delta u \cdot \nabla u.$$

Hence $(\Delta u)^2 \leq \sum_{j,k} (u_j u_k)_{jk} - 2 \nabla \Delta u \cdot \nabla u$. Therefore we have

$$\begin{aligned}
\int_{\mathbb{R}^n} (\Delta u)^2 g_2(\xi) &\leq \int_{\mathbb{R}^n} \sum_{j,k} (u_j u_k)_{jk} g_2(\xi) - 2 \int_{\mathbb{R}^n} \nabla \Delta u \cdot \nabla g_2(\xi) \\
&= \int_{\mathbb{R}^n} \sum_{j,k} u_j u_k g_2(\xi)_{jk} - 2 \int_{\mathbb{R}^n} \nabla \Delta u \cdot \nabla g_2(\xi) \\
&\leq C \int_{\mathbb{R}^n} |\nabla u|^2 g_1(\xi) + \delta \int_{\mathbb{R}^n} |\nabla \Delta u|^2 \xi^{6m} + C(\delta) \int_{\mathbb{R}^n} |\nabla u|^2 g_1(\xi) \\
&\leq C \int_{\mathbb{R}^n} |\nabla u|^2 g_1(\xi) + \delta \int_{\mathbb{R}^n} |\nabla \Delta u|^2 \xi^{6m}.
\end{aligned} \tag{5.11}$$

Combining with (5.10) and (5.11), by selecting the positive parameter ε small enough, we can obtain that

$$\int_{\mathbb{R}^n} |\nabla u|^2 g_1(\xi) + \int_{\mathbb{R}^n} (\Delta u)^2 g_2(\xi) \leq C \int_{\mathbb{R}^n} u^2 g_0(\xi) + \delta \int_{\mathbb{R}^n} |\nabla \Delta u|^2 \xi^{6m}.$$

By combining the above inequalities with (5.7) and selecting the positive parameter δ small enough, we have that

$$\int_{\mathbb{R}^n} |\nabla \Delta u|^2 \xi^{6m} + \int_{\mathbb{R}^n} |u|^{p+1} \xi^{6m} \leq C \int_{\mathbb{R}^n} u^2 g_0(\xi). \tag{5.12}$$

This proves (5.5). Further, we let $\xi = 1$ in B_R and $\xi = 0$ in B_{2R}^C , satisfying $|\nabla \xi| \leq \frac{C}{R}$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla \Delta u|^2 \xi^{6m} + \int_{\mathbb{R}^n} |u|^{p+1} \xi^{6m} &\leq C \int_{\mathbb{R}^n} u^2 g_0(\xi) \leq C R^{-6} \int_{\mathbb{R}^n} u^2 \xi^{6m-6} \\
&\leq C R^{-6} \left(\int_{\mathbb{R}^n} |u|^{p+1} \xi^{(3m-3)(p+1)} \right)^{\frac{2}{p+1}} R^{n(1-\frac{2}{p+1})}.
\end{aligned} \tag{5.13}$$

By selecting $m > 1$ and letting m close to 1, we can make sure that $(3m-3)(p+1) \leq 6m$. It follows that (5.6) holds. \square

5.2 Blow-down analysis and the proof of Theorem 1.1

The proof of Theorem 1.1. Firstly, we consider $1 < p \leq \frac{n+6}{n-6}$. If $p < \frac{n+6}{n-6}$, we can let $R \rightarrow \infty$ in (5.6) to get $u \equiv 0$ directly. However, if $p = \frac{n+6}{n-6}$, this gives

$$\int_{\mathbb{R}^n} |\nabla \Delta u|^2 + |u|^{p+1} < +\infty.$$

Hence

$$\lim_{R \rightarrow +\infty} \int_{B_{2R}(x) \setminus B_R(x)} |\nabla \Delta u|^2 + |u|^{p+1} = 0.$$

Then by (5.6), and noting that now $n = 6\frac{p+1}{p-1}$, we have

$$\begin{aligned} \int_{B_R(x)} |\nabla \Delta u|^2 + |u|^{p+1} &\leq CR^{-6} \int_{B_{2R}(x) \setminus B_R(x)} u^2 \\ &\leq CR^{-6} \left(\int_{B_{2R}(x) \setminus B_R(x)} |u|^{p+1} \right)^{\frac{2}{p+1}} R^{n(1-\frac{2}{p+1})} \leq C \left(\int_{B_{2R}(x) \setminus B_R(x)} |u|^{p+1} \right)^{\frac{2}{p+1}}. \end{aligned}$$

Let $R \rightarrow +\infty$, we get that $u \equiv 0$.

Secondly, we consider the supercritical case, i.e., $p > \frac{n+6}{n-6}$. We complete the proof via a few steps.

Step 1. $\lim_{\lambda \rightarrow \infty} E(u, 0, \lambda) < \infty$.

From Theorem 2.2 we know that E is nondecreasing w. r. t. λ , so we only need to show that $E(u, 0, \lambda)$ is bounded. Note that

$$E(u, 0, \lambda) \leq \frac{1}{\lambda} \int_{\lambda}^{2\lambda} E(u, 0, t) dt \leq \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} \int_t^{t+\lambda} E(u, 0, \gamma) d\gamma dt.$$

From Lemma 5.2, we have that

$$\frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} \int_t^{t+\lambda} \gamma^{6\frac{p+1}{p-1}-n} \left[\int_{B_{\gamma}} \frac{1}{2} |\nabla \Delta u|^2 dx - \frac{1}{p+1} \int_{B_{\gamma}} |u|^{p+1} dx \right] d\gamma dt \leq C,$$

where $C > 0$ is independent of γ .

$$\begin{aligned} &\frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} \int_t^{t+\lambda} \int_{\partial B_{\gamma}} \gamma^{6\frac{p+1}{p-1}-n-5} \left[\frac{6}{p-1} \left(\frac{6}{p-1} - 1 \right) \left(\frac{6}{p-1} - 2 \right) u \right. \\ &\quad \left. + \frac{18}{p-1} \left(\frac{6}{p-1} - 1 \right) \gamma \partial_r u + \frac{18}{p-1} \gamma^2 \partial_{rr} u + \gamma^3 \partial_{rrr} u \right] \\ &\quad \left[\frac{6}{p-1} \left(\frac{6}{p-1} - 1 \right) u + \frac{12}{p-1} \gamma \partial_r u + \gamma^2 \partial_{rr} u \right] \\ &\leq C \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} \int_t^{t+\lambda} t^{6\frac{p+1}{p-1}-n-5} \int_{\partial B_{\gamma}} [u^2 + \gamma^2 (\partial_r u)^2 + \gamma^4 (\partial_{rr} u)^2 + \gamma^6 (\partial_{rrr} u)^2] \\ &\leq C \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} t^{6\frac{p+1}{p-1}-n-5} \int_{B_{3\lambda}} [u^2 + \gamma^2 (\partial_r u)^2 + \gamma^4 (\partial_{rr} u)^2 + \gamma^6 (\partial_{rrr} u)^2] \\ &\leq C \lambda^{n-6\frac{p+1}{p-1}+6} \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} t^{6\frac{p+1}{p-1}-n-5} dt \\ &\leq C \end{aligned} \tag{5.14}$$

and

$$\begin{aligned}
& \left| \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} \int_t^{t+\lambda} \int_{\partial B_{\gamma}} \gamma^{6\frac{p+1}{p-1}-n-4} \frac{d}{d\gamma} (\gamma^2 \Delta u - \gamma^2 \partial_{rr} u - (n-1)\gamma \partial_r u)^2 \right| \\
& \leq \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} t^{6\frac{p+1}{p-1}-n-5} \int_t^{t+\lambda} \int_{\partial B_{\gamma}} |[2\gamma^2 \Delta u - 2\gamma^2 \partial_{rr} u - (n-1)\gamma \partial_r u] \\
& \quad [\gamma^2 \Delta u - \gamma^2 \partial_{rr} u - (n-1)\gamma \partial_r u]| \\
& \leq \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} t^{6\frac{p+1}{p-1}-n-5} \int_{B_{3\lambda}} |[2\gamma^2 \Delta u - 2\gamma^2 \partial_{rr} u - (n-1)\gamma \partial_r u] \\
& \quad [\gamma^2 \Delta u - \gamma^2 \partial_{rr} u - (n-1)\gamma \partial_r u]| \tag{5.15} \\
& \leq C \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} t^{6\frac{p+1}{p-1}-n-5} \int_{B_{3\lambda}} [\gamma^4 (\Delta u)^2 + \gamma^4 (\partial_{rr} u)^2 + \gamma^2 \partial_r u] \\
& \leq C \lambda^{n-6\frac{p+1}{p-1}+6} \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} t^{6\frac{p+1}{p-1}-n-5} dt \\
& \leq C.
\end{aligned}$$

The remaining terms can be treated similarly as the estimate (5.14) or (5.15).

Step 2. For any $\lambda > 0$, recall the definition

$$u^{\lambda}(x) := \lambda^{\frac{6}{p-1}} u(\lambda x),$$

and u^{λ} is also a smooth solution of (1.1) on \mathbb{R}^n . By rescaling (5.6), for $\lambda > 0$ and balls $B_r(x) \subset \mathbb{R}^n$,

$$\int_{B_r(x)} |\nabla \Delta u^{\lambda}|^2 + |u^{\lambda}|^{p+1} \leq C r^{n-6\frac{p+1}{p-1}}.$$

In particular, u^{λ} are uniformly bounded in $L_{loc}^{p+1}(\mathbb{R}^n)$ and $\nabla \Delta u^{\lambda}$ are uniformly bounded in $L_{loc}^2(\mathbb{R}^n)$. By elliptic estimates, u^{λ} are also uniformly bounded in $W_{loc}^{3,2}(\mathbb{R}^n)$. Hence up to a subsequence of $\lambda \rightarrow +\infty$, we can assume that $u^{\lambda} \rightarrow u^{\infty}$ weakly in $W_{loc}^{3,2}(\mathbb{R}^n) \cap L_{loc}^{p+1}(\mathbb{R}^n)$. By Sobolev embedding, $u^{\lambda} \rightarrow u^{\infty}$ strongly in $W_{loc}^{2,2}(\mathbb{R}^n)$. Then for any ball $B_R(0)$, by interpolation and noting (5.6), for any $q \in [1, p+1)$ as $\lambda \rightarrow +\infty$,

$$\|u^{\lambda} - u^{\infty}\|_{L^q(B_R(0))} \leq \|u^{\lambda} - u^{\infty}\|_{L^1(B_R(0))}^t \|u^{\lambda} - u^{\infty}\|_{L^{p+1}(B_R(0))}^{1-t} \rightarrow 0, \tag{5.16}$$

where $t \in (0, 1]$ satisfies $\frac{1}{q} = t + \frac{1-t}{p+1}$. That is, $u^{\lambda} \rightarrow u^{\infty}$ in $L_{loc}^q(\mathbb{R}^n)$ for any $q \in [1, p+1)$. For any function $\phi \in C_0^{\infty}(\mathbb{R}^n)$, we have that

$$\begin{aligned}
\int_{\mathbb{R}^n} \nabla \Delta u^{\infty} \cdot \nabla \Delta \phi - (u^{\infty})^{p-1} u^{\infty} \phi &= \lim_{\lambda \rightarrow +\infty} \int_{\mathbb{R}^n} \nabla \Delta u^{\lambda} \cdot \nabla \Delta \phi - (u^{\lambda})^{p-1} u^{\infty} \phi, \\
\int_{\mathbb{R}^n} |\nabla \Delta \phi|^2 - p(u^{\infty})^{p-1} \phi^2 &= \lim_{\lambda \rightarrow +\infty} \int_{\mathbb{R}^n} |\nabla \Delta \phi|^2 - p(u^{\lambda})^{p-1} \phi^2.
\end{aligned}$$

Therefore $u^\infty \in W_{loc}^{3,2}(\mathbb{R}^n) \cap L_{loc}^{p+1}(\mathbb{R}^n)$ is a stable solution of (1.1) in \mathbb{R}^n .

Step 3. The function u^∞ is homogeneous. Due to the scaling invariance of the functional E (i.e., $E(u, 0, R\lambda) = E(u^\lambda, 0, R)$) and the monotonicity formula, for any given $R_2 > R_1 > 0$, we see that

$$\begin{aligned} 0 &= \lim_{i \rightarrow \infty} (E(u, 0, R_2 \lambda_i) - E(u, 0, R_1 \lambda_i)) \\ &= \lim_{i \rightarrow \infty} (E(u^{\lambda_i}, 0, R_2) - E(u^{\lambda_i}, 0, R_1)) \\ &\geq C(n, p) \liminf_{i \rightarrow \infty} \int_{B_{R_2} \setminus B_{R_1}} r^{6\frac{p+1}{p-1}-n-6} \left(\frac{6}{p-1} u^{\lambda_i} + r \frac{\partial u^{\lambda_i}}{\partial r} \right)^2 dy dx \\ &\geq C(n, p) \int_{B_{R_2} \setminus B_{R_1}} r^{6\frac{p+1}{p-1}-n-6} \left(\frac{6}{p-1} u^\infty + r \frac{\partial u^\infty}{\partial r} \right)^2 dy dx. \end{aligned}$$

In the last inequality we have used the weak convergence of the sequence (u^{λ_i}) to the function u^∞ in $W_{loc}^{1,2}(\mathbb{R}^n)$ as $i \rightarrow \infty$. This implies that

$$\frac{6}{p-1} \frac{u^\infty}{r} + \frac{\partial u^\infty}{\partial r} = 0 \quad \text{a.e. in } \mathbb{R}^n.$$

Integrating in r shows that

$$u^\infty(x) = |x|^{-\frac{6}{p-1}} u^\infty\left(\frac{x}{|x|}\right).$$

That is, u^∞ is homogeneous.

Step 4. $u^\infty = 0$. This is a direct consequence of Theorem 4.1. Since this holds for the limit of any sequence $\lambda \rightarrow +\infty$, by (5.16) we get

$$\lim_{\lambda \rightarrow +\infty} u^\lambda \quad \text{strongly in } L^2(B_4(0)).$$

Step 5. $u = 0$. For all $\lambda \rightarrow +\infty$, we see that

$$\lim_{\lambda \rightarrow +\infty} \int_{B_4(0)} (u^\lambda)^2 = 0.$$

By (5.6),

$$\lim_{\lambda \rightarrow +\infty} \int_{B_3(0)} |\nabla \Delta u^\lambda|^2 + |u^\lambda|^{p+1} \leq \lim_{\lambda \rightarrow +\infty} \int_{B_4(0)} (u^\lambda)^2 = 0. \quad (5.17)$$

By the elliptic interior L^2 estimate, we get that

$$\lim_{\lambda \rightarrow +\infty} \int_{B_2(0)} \sum_{k \leq 3} |\nabla^k u^\lambda|^2 = 0.$$

In particular, we can choose a sequence $\lambda_i \rightarrow +\infty$ such that

$$\int_{B_2(0)} \sum_{k \leq 3} |\nabla^k u^{\lambda_i}|^2 \leq 2^{-i}.$$

Hence we have

$$\int_1^2 \sum_{i=1}^{+\infty} \int_{\partial B_r} \sum_{k \leq 3} |\nabla^k u^{\lambda_i}|^2 dr \leq \sum_{i=1}^{+\infty} \int_1^2 \int_{\partial B_r} \sum_{k \leq 3} |\nabla^k u^{\lambda_i}|^2 dr \leq 1.$$

That is, the function

$$H(r) := \sum_{i=1}^{+\infty} \int_{\partial B_r} \sum_{k \leq 3} |\nabla^k u^{\lambda_i}|^2 \in L^1(1, 2).$$

Then there exists an $r_0 \in (1, 2)$ such that $H(r_0) < +\infty$, by which we get that

$$\lim_{i \rightarrow +\infty} \|u^{\lambda_i}\|_{W^{3,2}(\partial B_{r_0})} = 0.$$

Combining this with (5.17) and the scaling invariance of $E(r)$, we have

$$\lim_{i \rightarrow +\infty} E(\lambda_i r_0, 0, u) = \lim_{i \rightarrow +\infty} E(r_0, 0, u^{\lambda_i}) = 0.$$

Since $\lambda_i r_0 \rightarrow +\infty$ and $E(r, 0, u)$ is non-decreasing in r , we get

$$\lim_{r \rightarrow +\infty} E(r, 0, u) = 0.$$

By the smoothness of u , $\lim_{r \rightarrow 0} E(r, 0, u) = 0$. Then again by the monotonicity of $E(r, 0, u)$ and step 4, we obtain that

$$E(r, 0, u) = 0 \quad \text{for all } r > 0.$$

Therefore, by the monotonicity formula we know that u is homogeneous, then $u \equiv 0$ by Theorem 4.1. \square

6 Finite Morse index solutions

In this section, we prove Theorem 1.2. We always assume that u is a smooth classical solution with finite Morse index.

Lemma 6.1. *Let u be a smooth solution (positive or sign changing) of (1.1) with finite Morse index, then there exist constants $C > 0$ and R_0 such that*

$$|u(x)| \leq C|x|^{-\frac{6}{p-1}}, \quad \forall x \in B_{R_0}(0)^c.$$

Proof. Since that u is stable outside B_{R_0} . For $x \in B_{R_0}^c$, let $M(x) = |u(x)|^{\frac{p-1}{6}}$ and $d(x) = |x| - R_0$. Assume that there exists a sequence of $x_k \in B_{R_0}^c$ such that

$$M(x_k)d(x_k) \geq 2k. \quad (6.1)$$

Since u is bounded on any compact set of \mathbb{R}^n , $d(x_k) \rightarrow +\infty$.

By the doubling Lemma [25], there exists another sequence $y_k \in B_{R_0}^c$, such that

$$M(y_k)d(y_k) \geq 2k, \quad M(y_k) \geq M(x_k);$$

$$M(z) \leq 2M(y_k) \quad \text{for any } z \in B_{R_0}^c \quad \text{such that } |z - y_k| \leq \frac{k}{M(y_k)}.$$

Now we define

$$u_k(x) := M(y_k)^{-\frac{6}{p-1}} u(y_k + M(y_k)^{-1}x), \quad \text{for } x \in B_k(0).$$

This and above arguments give that, $u_k(0) = 1$, $|u_k| \leq 2^{\frac{6}{p-1}}$ in $B_k(0)$. Further, $B_{k/M(y_k)} \cap B_{R_0} = \emptyset$, which implies that u is a stable solution in $B_{k/M(y_k)}(y_k)$. Hence, u_k is stable in $B_k(0)$.

By elliptic regularity theory, u_k are uniformly bounded in $C_{loc}^7(B_k(0))$. Up to a sequence, u_k convergence to u_∞ in $C_{loc}^6(\mathbb{R}^n)$. By the above conditions on u_k , we have

$$|u_\infty(0)| = 1, \quad |u_\infty| \leq 2^{\frac{6}{p-1}}; \quad u_\infty \text{ is a smooth stable solution of (1.1) in } \mathbb{R}^n. \quad (6.2)$$

By the Liouville theorem for stable solution, we have $u_\infty \equiv 0$, a contradiction with (6.1). \square

Corollary 6.1. *There exist constants $C > 0$ and R_0 such that for all $x \in B_{R_0}^c$,*

$$\sum_{k \leq 5} |x|^{\frac{6}{p-1}+k} |\nabla^k u(x)| \leq C. \quad (6.3)$$

Proof. For any x_0 with $|x_0| > 3R_0$, take $\lambda = \frac{|x_0|}{2}$ and define

$$\bar{u}(x) = \lambda^{\frac{6}{p-1}} u(x_0 + \lambda x).$$

By the previous Lemma, $\bar{u} \leq C$ in $B_1(0)$. By the elliptic regularity theory we have

$$\sum_{k \leq 5} |\nabla^k \bar{u}(0)| \leq C.$$

Scaling back we get the conclusions. \square

6.1 The proof of Theorem 1.2-(1): $1 < p < \frac{n+6}{n-6}$ (Subcritical case)

We need the following Pohozaev identity. A general version can be seen in [27].

Lemma 6.2. *For any function u satisfying (1.1), we have that*

$$\left(\frac{n-6}{2} - \frac{n}{p+1}\right) \int_{B_R} |u|^{p+1} = \int_{\partial B_R} B_3(u) d\sigma,$$

where

$$\begin{aligned} -B_3(u) = & \frac{R}{p+1} |u|^{p+1} - 2(-\Delta)^2 u \frac{\partial u}{\partial n} + 2u \frac{\partial(-\Delta)^2 u}{\partial n} - \frac{R}{2} |\nabla \Delta u|^2 \\ & - \frac{n-2}{2} u \frac{\partial \Delta^2 u}{\partial n} + \frac{n-2}{2} \Delta u \frac{\partial \Delta u}{\partial n} + \langle x \cdot \nabla u \rangle \frac{\partial \Delta^2 u}{\partial n} \\ & - \Delta^2 u \frac{\partial \langle x \cdot \nabla u \rangle}{\partial n} + \langle x \cdot \nabla \Delta u \rangle \frac{\partial \Delta u}{\partial n}. \end{aligned} \quad (6.4)$$

The proof of Theorem 1.2-(1). By Corollary 6.1, for $R > R_0$ (R_0 is defined in Corollary 6.1), noting that $p < \frac{n+6}{n-6}$ (hence $n - 6\frac{p+1}{p-1} < 0$), we have the following estimate

$$\int_{\partial B_R} |B_3(u)| d\sigma \leq C \int_{\partial B_R} R^{-\frac{12}{p-1}-5} d\sigma \leq C R^{n-6\frac{p+1}{p-1}} \rightarrow 0 \text{ as } R \rightarrow +\infty.$$

Combining with the Pohazaev identity, letting $R \rightarrow +\infty$, we get that

$$\left(\frac{n-6}{2} - \frac{n}{p+1}\right) \int_{\mathbb{R}^n} |u|^{p+1} = 0.$$

Since $\frac{n-6}{2} - \frac{n}{p+1} < 0$, we obtain that $u \equiv 0$.

6.2 The proof of Theorem 1.2-(3): $p = \frac{n+6}{n-6}$ (critical case)

Since u is stable outside B_{R_0} , Lemma 5.2 still holds if the support of η is outside B_{R_0} . Take $\varphi \in C_0^\infty(B_{2R} \setminus B_{2R_0})$ such that $\varphi \equiv 1$ in $B_R \setminus B_{3R_0}$ and $\sum_{k \leq 5} |x|^k |\nabla^k u| \leq 1000$. Then by choosing $\eta = \varphi^m$, where m is bigger than 1, we get that

$$\int_{B_R \setminus B_{3R_0}} |\nabla \Delta u|^2 + |u|^{p+1} \leq C.$$

Letting $R \rightarrow +\infty$, we have

$$\int_{\mathbb{R}^n} |\nabla \Delta u|^2 + |u|^{p+1} < +\infty. \quad (6.5)$$

By the interior elliptic estimates and Holder's inequality, we have

$$\begin{aligned} R^{-4} \int_{B_{2R} \setminus B_R} |\nabla u|^2 & \leq C \int_{B_{3R} \setminus B_{R/2}} |\nabla \Delta u|^2 + C \left(\int_{B_{3R} \setminus B_{R/2}} |u|^{p+1} \right)^{\frac{2}{p+1}}, \\ R^{-2} \int_{B_{2R} \setminus B_R} |\Delta u|^2 & \leq C \int_{B_{3R} \setminus B_{R/2}} |\nabla \Delta u|^2 + C \left(\int_{B_{3R} \setminus B_{R/2}} |u|^{p+1} \right)^{\frac{2}{p+1}}, \\ R^{-6} \int_{B_{2R} \setminus B_R} u^2 & \leq C \int_{B_{3R} \setminus B_{R/2}} |\nabla \Delta u|^2 + C \left(\int_{B_{3R} \setminus B_{R/2}} |u|^{p+1} \right)^{\frac{2}{p+1}}. \end{aligned}$$

Therefore, we have that

$$\max \left(R^{-4} \int_{B_{2R} \setminus B_R} |\nabla u|^2, R^{-2} \int_{B_{2R} \setminus B_R} |\Delta u|^2, R^{-6} \int_{B_{2R} \setminus B_R} u^2 \right) \rightarrow 0$$

as $R \rightarrow +\infty$. On the other hand, testing (1.1) with an compact support function η^2 , we get

$$\int_{\mathbb{R}^n} |\nabla \Delta u|^2 \eta^2 - |u|^{p+1} \eta^2 = - \int_{\mathbb{R}^n} \nabla \Delta u \cdot \nabla \Delta \eta^2 \cdot u + \nabla \Delta u \nabla u \Delta \eta^2 + \nabla \Delta u \nabla (2 \nabla u \nabla \eta^2).$$

By selecting $\eta(x) = \xi(\frac{x}{R})^{3m}$, $m > 1$ and $\xi \in C_0^\infty(B_2)$ and $\xi \equiv 1$ in B_1 , and $\sum_{k \leq 3} |\nabla^k u| \leq 1000$, we get that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} |\nabla \Delta u|^2 \xi \left(\frac{x}{R} \right)^{6m} - |u|^{p+1} \xi \left(\frac{x}{R} \right)^{6m} \right| &\leq C \left(R^{-4} \int_{B_{2R} \setminus B_R} |\nabla u|^2 \right. \\ &\quad \left. + R^{-2} \int_{B_{2R} \setminus B_R} |\Delta u|^2 + R^{-6} \int_{B_{2R} \setminus B_R} u^2 \right). \end{aligned}$$

Now letting $R \rightarrow +\infty$, we obtain that

$$\int_{\mathbb{R}^n} |\nabla \Delta u|^2 - |u|^{p+1} = 0.$$

Combining with (6.5), we get the conclusions.

6.3 The proof of Theorem 1.2-(2): $p > \frac{n+6}{n-6}$ (supercritical case)

Lemma 6.3. *There exists a constant $C > 0$ such that $E(r, 0, u) \leq C$ for all $r > 3R_0$.*

Proof. From the monotonicity formula, combine the derivative estimates (6.3), we have the following estimates

$$\begin{aligned} E(r, 0, u) &\leq C r^{4\frac{n+6}{n-6}-n} \left(\int_{B_r} |\nabla \Delta u|^2 + |u|^{p+1} \right) \\ &\quad + \sum_{j,k \leq 4, j+k \leq 5} r^{6\frac{n+1}{p-1}-n-5+j+k} \int_{\partial B_r} |\nabla^j u| |\nabla^k u| \\ &\leq C. \end{aligned}$$

This constant only depends on the constant in (6.3). □

As a consequence, we have the following

Corollary 6.2.

$$\int_{B_{3R_0}^c} \frac{(\frac{6}{p-1}u(x) + |x|u_r(x))^2}{|x|^{n+6-6\frac{n+6}{n-6}}} dx < +\infty. \quad (6.6)$$

As before, we define the blowing down sequence

$$u^\lambda(x) = \lambda^{\frac{6}{p-1}} u(\lambda x).$$

By Lemma 6.1, u^λ are uniformly bounded in $C^7(B_r(0) \setminus B_{1/r}(0))$ for any fixed $r > 1$ and moreover, u^λ is stable outside $B_{R_0/\lambda}(0)$. There exists a function $u^\infty \in C^6(\mathbb{R}^n \setminus \{0\})$, such that up to a subsequence of $\lambda \rightarrow +\infty$, u^λ converges to u^∞ in $C_{loc}^6(\mathbb{R}^n \setminus \{0\})$, u^∞ is stable solution of (1.1) in $\mathbb{R}^n \setminus \{0\}$.

For any $r > 1$, by the above Corollary 6.6,

$$\begin{aligned} & \int_{B_r \setminus B_{1/r}} \frac{(\frac{6}{p-1} u^\infty(x) + |x| u_r^\infty(x))^2}{|x|^{n+6-6\frac{n+6}{n-6}}} dx \\ &= \lim_{\lambda \rightarrow +\infty} \int_{B_r \setminus B_{1/r}} \frac{(\frac{6}{p-1} u^\lambda(x) + |x| u_r^\lambda(x))^2}{|x|^{n+6-6\frac{n+6}{n-6}}} dx \\ &= \lim_{\lambda \rightarrow +\infty} \int_{B_{\lambda r} \setminus B_{\lambda/r}} \frac{(\frac{6}{p-1} u(x) + |x| u_r(x))^2}{|x|^{n+6-6\frac{n+6}{n-6}}} dx \\ &= 0. \end{aligned}$$

Hence, u^∞ is homogeneous, and by Theorem 4.1, $u^\infty \equiv 0$ if $p < p_c(n)$. Since this hold for any limit of u^λ as $\lambda \rightarrow +\infty$, then we have

$$\lim_{|x| \rightarrow +\infty} |x|^{\frac{6}{p-1}} |u(x)| = 0.$$

Then as in the proof of Corollary 6.1, we have

$$\lim_{|x| \rightarrow +\infty} \sum_{k \leq 6} |x|^{\frac{6}{p-1}+k} |\nabla^k u(x)| = 0.$$

Therefore, for any $\varepsilon > 0$, take an R_0 such that for $|x| > R_0$, there holds

$$\sum_{k \leq 6} |x|^{\frac{6}{p-1}+k} |\nabla^k u(x)| \leq \varepsilon.$$

Then for any $r \gg R_0$, we have

$$\begin{aligned} E(r; 0, u) &\leq C r^{6\frac{p+1}{p-1}-n} \int_{B_R(0)} (|\nabla \Delta u|^2 + |u|^{p+1}) \\ &\quad + C \varepsilon r^{6\frac{p+1}{p-1}-n} \int_{B_r(0) \setminus B_R(0)} |x|^{-6\frac{p+1}{p-1}} + C \varepsilon r^{6\frac{p+1}{p-1}+1-n} \int_{\partial B_r(0)} |x|^{-6\frac{p+1}{p-1}} \\ &\leq C(R_0)(r^{6\frac{p+1}{p-1}-n} + \varepsilon). \end{aligned}$$

We obtain that $\lim_{r \rightarrow +\infty} E(r; 0, u) = 0$ since $6\frac{p+1}{p-1} - n < 0$ and ε can be arbitrarily small. On the other hand, since u is smooth we have $\lim_{r \rightarrow 0} E(r; 0, u) = 0$, thus $E(r; 0, u) = 0$ for all $r > 0$, thus by the monotonicity formula we get that u is homogeneous, and then by Theorem 4.1, we know that $u \equiv 0$.

7 Proofs of Theorem 2.2

To further investigate the optimal condition to make the monotonicity formula hold, we find that we have drop the term $\int_{\partial B_1} \lambda (\frac{dv^\lambda}{d\lambda})^2$. Recall that

$$\begin{aligned} v^\lambda &= \Delta u^\lambda = \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} + (n-1 - \frac{12}{p-1}) \lambda \frac{du^\lambda}{d\lambda} + \frac{6}{p-1} (\frac{6}{p-1} - n + 2) u^\lambda + \Delta_\theta u^\lambda \\ &:= \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} + a \lambda \frac{du^\lambda}{d\lambda} + b u^\lambda + \Delta_\theta u^\lambda. \end{aligned}$$

By some integrate by part, we have that

$$\begin{aligned} \int_{\partial B_1} \lambda (\frac{dv^\lambda}{d\lambda})^2 &= \int_{\partial B_1} \left(\lambda^5 (\frac{d^3 u^\lambda}{d\lambda^3})^2 + (a^2 - 2a - 2b - 4) \lambda^3 (\frac{d^2 u^\lambda}{d\lambda^2})^2 \right. \\ &\quad \left. + (-a^2 + b^2 + 2a + 2b) \lambda (\frac{du^\lambda}{d\lambda})^2 \right) \\ &+ \int_{\partial B_1} \left(-2\lambda^3 (\nabla_\theta \frac{d^2 u^\lambda}{d\lambda^2})^2 + (10 - 2b) \lambda (\nabla_\theta \frac{du^\lambda}{d\lambda})^2 \right) + \int_{\partial B_1} \lambda (\Delta_\theta \frac{du^\lambda}{d\lambda})^2 \\ &+ \frac{d}{d\lambda} \left(\int_{\partial B_1} \sum_{0 \leq i, j \leq 2, i+j \leq 2} c_{i,j}^1 \lambda^{i+j} \frac{d^i u^\lambda}{d\lambda^i} \frac{d^j u^\lambda}{d\lambda^j} + \sum_{0 \leq s, t \leq 2, s+t \leq 2} c_{s,t}^2 \lambda^{s+t} \frac{d^s u^\lambda}{d\lambda^s} \frac{d^t u^\lambda}{d\lambda^t} \right) \end{aligned}$$

where $c_{i,j}^1, c_{s,t}^2$ determined by a, b hence by p, n . From Theorem 2.1 we derive that

$$\begin{aligned} \frac{dE^c(\lambda, x, u)}{d\lambda} &= \int_{\partial B_1} 3\lambda^5 \left(\frac{d^3 u^\lambda}{d\lambda^3} \right)^2 + A_1 \lambda^3 \left(\frac{d^2 u^\lambda}{d\lambda^2} \right)^2 + A_2 \lambda \left(\frac{du^\lambda}{d\lambda} \right)^2 \\ &+ \int_{\partial B_1} \left(2\lambda^3 (\nabla_\theta \frac{d^2 u^\lambda}{d\lambda^2})^2 + (8\alpha - 4\beta - 2b + 4n - 18) \lambda (\nabla_\theta \frac{du^\lambda}{d\lambda})^2 \right) \\ &+ \int_{\partial B_1} \lambda (\Delta_\theta \frac{du^\lambda}{d\lambda})^2, \end{aligned} \quad (7.1)$$

where

$$\begin{aligned} A_1 &:= 10\delta_1 - 2\delta_2 - 56 + a^2 - 2a - 2b - 4 \\ A_2 &:= -18\delta_1 + 6\delta_2 - 4\delta_3 + 2\delta_4 + 72 - a^2 + b^2 + 2a + 2b, \end{aligned} \quad (7.2)$$

Let $k := \frac{6}{p-1}$, a direct calculation we have that

$$\begin{aligned} A_1 &= -10k^2 + (-60 + 10n)k - n^2 + 24n - 83, \\ A_2 &= 3k^4 + (36 - 6n)k^3 + (3n^2 - 48n + 150)k^2 \\ &\quad + (12n^2 - 114n + 252)k + 9n^2 - 72n + 135, \end{aligned}$$

and

$$B_1 := 8\alpha - 4\beta - 2b + 4n - 18 = -6k^2 + (-36 + 6n)k + 12n - 42.$$

Notice that our supercritical condition $p > \frac{n+6}{n-6}$ is equivalent to $0 < k < \frac{n-6}{2}$. Firstly, we have the following lemma which yields the sign of A_2 and B_1 .

Lemma 7.1. *If $p > \frac{n+6}{n-6}$, then $A_2 > 0$ and $B_1 > 0$.*

Proof. From (7.2), we derive that

$$A_2 = 3(k+1)(k+3)(k-(n-5))(k-(n-3)), \quad (7.3)$$

and the roots of $B_1 = 0$ are

$$\frac{1}{2}n - 3 - \frac{1}{2}\sqrt{n^2 - 4n + 8}, \frac{1}{2}n - 3 + \frac{1}{2}\sqrt{n^2 - 4n + 8}$$

Recall that $p > \frac{n+6}{n-6}$ is equivalent to $0 < k < \frac{n-6}{2}$, we get the conclusion. \square

To show monotonicity formula, we proceed it to prove the following inequality

$$3\lambda^5 \left(\frac{d^3 u^\lambda}{d\lambda^3} \right)^2 + A_1 \lambda^3 \left(\frac{d^2 u^\lambda}{d\lambda^2} \right)^2 + A_2 \lambda \left(\frac{du^\lambda}{d\lambda} \right)^2 \quad (7.4)$$

$$\geq \epsilon \lambda \left(\frac{du^\lambda}{d\lambda} \right)^2 + \frac{d}{d\lambda} \left(\sum_{0 \leq i, j \leq 2} c_{i,j} \lambda^{i+j} \frac{d^i u^\lambda}{d\lambda^i} \frac{d^j u^\lambda}{d\lambda^j} \right). \quad (7.5)$$

To deal with the rest of the dimensions, we employ the second idea: we find nonnegative constants d_1, d_2 and constants c_1, c_2 such that we have the following Jordan form decomposition:

$$\begin{aligned} 3\lambda^5 (f''')^2 + A_1 \lambda^3 (f'')^2 + A_2 \lambda (f')^2 &= 3\lambda (\lambda^2 f''' + c_1 \lambda f'')^2 + d_1 \lambda (\lambda f'' + c_2 f')^2 \\ &+ d_2 \lambda (f')^2 + \frac{d}{d\lambda} \left(\sum_{i,j} e_{i,j} \lambda^{i+j} f^{(i)} f^{(j)} \right), \end{aligned} \quad (7.6)$$

where the unknown constants are to be determined.

Lemma 7.2. *Let $p > \frac{n+6}{n-6}$ and A_1 satisfy*

$$A_1 + 12 > 0, \quad (7.7)$$

then there exist nonnegative numbers d_1, d_2 , and real numbers $c_1, c_2, e_{i,j}$ such that the differential inequality (7.6) holds.

Proof. Since

$$4\lambda^4 f''' f'' = \frac{d}{d\lambda} (2\lambda^4 (f'')^2) - 8\lambda^3 (f'')^2$$

and

$$2\lambda^2 f'' f' = \frac{d}{d\lambda} (\lambda^2 (f')^2) - 2\lambda (f')^2,$$

by comparing the coefficients of $\lambda^3 (f'')^2$ and $\lambda (f')^2$, we have that

$$d_1 = A_1 - 3c_1^2 + 12c_1, \quad d_2 = A_2 - (c_2^2 - 2c_2)(A_1 - 3c_1^2 + 12c_1).$$

In particular,

$$\max_{c_1} d_1(c_1) = A_1 + 12 \text{ and the critical point is } c_1 = 2.$$

Since $A_2 > 0$, we select that $c_1 = 2, c_2 = 0$. Hence, in this case, by a direct calculation we see that $d_1 = A_1 + 12 > 0$. Then we get the conclusion. \square

We conclude from Lemma 7.2 that if $A_1 + 12 > 0$ then (7.4) holds. This implies that when $7 \leq n \leq 20, p > \frac{n+6}{n-6}$ or $n \geq 21$ and

$$\frac{n+6}{n-6} < p < \frac{5n+30-\sqrt{15n^2-60n+190}}{5n-30-\sqrt{15n^2-60n+190}} \quad (7.8)$$

then (7.4) holds.

Combine idea from the above with the the following idea, we can get better condition to make the monotonicity formula holds. We start from the differential identity (7.6). Recall that the derivative term is a 'good' term since it can be absorbed in the term $E^c(\lambda, x, u)$.

We make use of two ideas to prove (7.4). The second idea is straightforward. We use the positivity of terms $A_2\lambda(\frac{du}{d\lambda})^2$ and $3\lambda^5(\frac{d^3u}{d\lambda^3})^2$ to bound the term $A_1(\frac{d^2u}{d\lambda^2})^2$. Note

$$\begin{aligned} 3\lambda(\lambda^2 f''' + 2\lambda f'')^2 + A_2\lambda(f')^2 &\geq -2\sqrt{3A_2}(\lambda^3 f''' f' + 2\lambda^2 f'' f') \\ &= 2\sqrt{3A_2}(\lambda^3(f'')^2 - \lambda(f')^2) + 2\sqrt{3A_2}\frac{d}{d\lambda}\left(\lambda^3 f'' f' - \frac{1}{2}\lambda^2(f')^2\right). \end{aligned}$$

We observe that We divide the term $A_2\lambda(f')^2$ into two parts, $\theta A_2(f')^2$ and $(1 - \theta)\lambda(f')^2$, and then, find the optimal parameter θ . Following this idea, we have

$$\begin{aligned} &3\lambda(\lambda^2 f''' + 2\lambda f'')^2 + (A_1 + 12)\lambda^3(f'')^2 + A_2\lambda(f')^2 \\ &\geq (A_1 + 12 + 2\sqrt{3\alpha A_2})\lambda^3(f'')^2 + \left((1 - \alpha)A_2 - 2\sqrt{3\alpha A_2}\right)\lambda(f')^2 \\ &\quad + 2\sqrt{3\alpha A_2}\frac{d}{d\lambda}\left(\lambda^3 f'' f' - \frac{1}{2}\lambda^2(f')^2\right), \end{aligned} \quad (7.9)$$

hence, we have the desired monotonicity formula once the following two inequalities hold:

$$A_1 + 12 + 2\sqrt{3\alpha A_2} \geq 0, \quad (1 - \alpha)A_2 - 2\sqrt{3\alpha A_2} > 0. \quad (7.10)$$

The second inequality of (7.10) gives the range of α , that is

$$\frac{12\alpha}{(\alpha - 1)^2} < \min_{0 \leq k \leq \frac{n-6}{2}} A_2. \quad (7.11)$$

Obviously, the first inequality of (7.10) holds if $A_1 + 12 \geq 0$. So we just need to check the following inequality:

$$A_1 + 12 > -\sqrt{12\alpha A_2} \text{ where } \frac{8\alpha}{(\alpha - 1)^2} < \min_{0 \leq k \leq \frac{n-6}{2}} A_2. \quad (7.12)$$

We discuss the remaining dimensions as follows:
When $n = 21$,

$$\min_{0 \leq k \leq \frac{n-6}{2} | n=14} A_2 = A_2(k=0) = 2592,$$

thus from (7.11) we get that $\alpha \leq 0.9342$, then $(A_1 + 12)^2 - 12\alpha |_{\alpha=0.9342} A_2 < 0$ if $-0.5941782055 < k < 4.483334837$. On the other hand, $A_1 + 12 < 0$ implies that $0 < k < 0.05352432355$. Hence, (7.12) holds.

The case of $21 \leq n \leq 30$ can be dealt with similarly. We omit the details here.

Let

$$p_m(n) := \begin{cases} +\infty & \text{if } n \leq 30, \\ \frac{5n+30-\sqrt{15n^2-60n+190}}{5n-30-\sqrt{15n^2-60n+190}} & \text{if } n \geq 31. \end{cases}$$

Combining all the lemmas of this section, we obtain the following theorem.

Theorem 7.1. For $\frac{n+6}{n-6} < p < p_m(n)$, then there exists a $C(n, p) > 0$ such that

$$\frac{d}{d\lambda} E^c(\lambda, x, u) \geq C(n, p) \int_{\partial B_1} \lambda \left(\frac{du^\lambda}{d\lambda} \right)^2.$$

Proof of Theorem 2.2. Let $d(n)$ be defined at (1.6) (See also the appendix). By Lemma 8.3 of the appendix, we know that $d(n) < \sqrt{n}$ for $n \geq 15$. Hence, we have the following inequalities

$$\frac{n-8}{2} - d(n) > \frac{n-8}{2} - \sqrt{n} \quad (7.13)$$

and

$$\frac{n-8}{2} - \sqrt{n} \geq \frac{1}{2}n - 3 - \frac{1}{10}\sqrt{15n^2 - 60n + 190} \quad \text{for } n \geq 14. \quad (7.14)$$

Hence we derive that when $n \geq 15$

$$\frac{n+4-2d(n)}{n-8-2d(n)} < \frac{5n+30-\sqrt{15n^2-60n+190}}{5n-30-\sqrt{15n^2-60n+190}}.$$

Therefore, $p_c(n) < p_m(n)$. Theorem 2.2 is thus proved. \square

8 Appendix: Proof of Lemma 4.1

In this appendix, we prove the technical lemma 4.1.

Recall the definition of c_0, c_1 and c_2 in (4.6). Let

$$k := \frac{n-8}{2} + a.$$

Then c_0 can be rewritten in terms of a :

$$\begin{aligned} c_0 = & -a^6 + (8 + \frac{3}{4}n^2)a^4 - (16 + \frac{3}{16}n^4)a^2 \\ & + \frac{3}{16}n^5 - \frac{15}{16}n^4 - \frac{3}{2}n^3 + \frac{33}{4}n^2 + 3n - 9. \end{aligned}$$

Further, we let $t := a^2$, we get a three-order algebraic equation as following:

$$\begin{aligned} c_0 = & -t^3 + (8 + \frac{3}{4}n^2)t^2 - (16 + \frac{3}{16}n^4)t \\ & + \frac{3}{16}n^5 - \frac{15}{16}n^4 - \frac{3}{2}n^3 + \frac{33}{4}n^2 + 3n - 9. \end{aligned} \quad (8.1)$$

By the two crucial transformation above, we reduce a six-order algebraic equation to a third order algebraic equation. Now we can get the explicit solution of the above equation (8.1) which has two imaginary roots and one real root. We denote the real root as $d(n)$. Let

$$d_1(n) := -94976 + 20736n + 103104n^2 - 10368n^3 + 1296n^5 - 3024n^4 - 108n^6,$$

$$\begin{aligned} d_2(n) := & 6131712 - 16644096n^2 + 6915840n^4 - 690432n^6 - 3039232n \\ & + 4818944n^3 - 1936384n^5 + 251136n^7 - 30864n^8 - 4320n^9 \\ & + 1800n^{10} - 216n^{11} + 9n^{12} \end{aligned}$$

and

$$d_0(n) := -(d_1(n) + 36\sqrt{d_2(n)})^{1/3}. \quad (8.2)$$

Notice that

$$\begin{aligned} d_2(n) := & (9n^8 - 216n^7 + 1872n^6 - 6048n^5 - 16032n^4 + 206208n^3 \\ & - 848640n^2 - 189952n + 383232)(n-2)^2(n+2)^2 > 0 \text{ if } n \geq 12, \end{aligned}$$

hence $\sqrt{d_2(n)}$ is well defined whenever $n \geq 12$. Define

$$d(n) := \frac{1}{6} \left(9n^2 + 96 - \frac{1536 + 1152n^2}{d_0(n)} - \frac{3}{2}d_0(n) \right)^{1/2}. \quad (8.3)$$

By the proof of Lemma 8.3 below, we will see that $d(n)$ is well-defined, i.e., $9n^2 + 96 > \frac{1536 + 1152n^2}{d_0(n)} + \frac{3}{2}d_0(n)$.

Let r_1, r_2 denote the two real roots of c_0 which can be computed as

$$r_1 := \frac{n-8}{2} - d(n), \quad r_2 := \frac{n-8}{2} + d(n). \quad (8.4)$$

Therefore, we see that $c_0 > 0$ whenever $r_1 < k < r_2$. Since the roots r_1 and r_2 depend on $d_0(n)$, we must have a fine estimate on $d_0(n)$.

Lemma 8.1. *The $d_0(n)$ has the following properties:*

(1)

$$d_0(n) := \frac{256(3n^2 + 4)}{(36\sqrt{d_2(n)} - d_1(n))^{1/3}}. \quad (8.5)$$

(2) For $n \geq 15$, then

$$\frac{d}{dn}d_0(n) < 0, \quad 128 < d_0(n) < 187.$$

Proof. The proof of (1) comes from the following identity which can be checked directly:

$$d_1^2(n) - 36^2 d_2^2(n) = 256^3 (3n^2 + 4)^3. \quad (8.6)$$

Now we start to prove the conclusion (2) of the Lemma. From (8.6) we see that $d_0(n) > 0$. Thus we only need to show that $\frac{d}{dn}d_0(n) < 0$. In fact,

$$\frac{d}{dn}d_0^3(n) = -\frac{d}{dn}d_1(n) - \frac{18\frac{d}{dn}d_2(n)}{\sqrt{d_2(n)}}, \quad (8.7)$$

since

$$-\frac{d}{dn}d_1(n) = 648n^5 - 6480n^4 + 12096n^3 + 31104n^2 - 206208n - 20736 > 0 \quad \text{if } n \geq 8,$$

and furthermore

$$\begin{aligned} & \left(-\frac{d}{dn}d_1(n)\right)^2 \cdot d_2(n) - 18^2 \left(\frac{d}{dn}d_2(n)\right)^2 \\ &= -293534171136n^{15} + 4109478395904n^{14} - 9001714581504n^{13} \\ & \quad - 168292924784640n^{12} + 1233104438034432n^{11} - 3119550711201792n^{10} \\ & \quad - 6748415824232448n^9 + 21348066225291264n^8 - 1783991975804928n^7 \\ & \quad + 9835612546793472n^6 + 34945090870837248n^5 - 114643053771227136n^4 \\ & \quad + 19014404334944256n^3 - 110880250103070720n^2 - 14427791579676672n \\ & \quad - 356241767399424 \\ &= -10871635968(3n^3 - 18n^2 + 84n + 8)(n^4 - 8n^3 - 40n^2 + 480n + 16) \\ & \quad \cdot (n-2)^2(n+2)^2(3n^2+4)^2 < 0 \quad \text{if } n \geq 3. \end{aligned}$$

Thus, combining the above two equations, we get that

$$-\frac{d}{dn}d_1(n) \cdot \sqrt{d_2(n)} < 18\frac{d}{dn}d_2(n).$$

Hence, if we combine this with (8.7), we have that

$$\frac{d}{dn}d_0^3(n) = -\frac{d}{dn}d_1(n) - \frac{18\frac{d}{dn}d_2(n)}{\sqrt{d_2(n)}} < 0.$$

Therefore, $\frac{d}{dn}d_0(n) < 0$ for $n \geq 15$. Notice that $d_0(n)|_{n=15} \simeq 186.0929 < 187$ and a straightforward calculation shows that

$$\lim_{n \rightarrow +\infty} d_0(n) = 128.$$

By the monotonicity of $d_0(n)$ for $n \geq 15$, we derive that $d_0(n) \in (128, 187)$ for $n \geq 15$. □

By straightforward calculation we have the following asymptotic properties.

Lemma 8.2.

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{d(n)}{\sqrt{n}} &= 1, \quad \lim_{n \rightarrow +\infty} (d(n) - \sqrt{n}) = 0, \\ \lim_{n \rightarrow +\infty} \sqrt{n}(d(n) - \sqrt{n}) &= -\frac{1}{2}. \end{aligned}$$

By Lemma 8.2 above, we known that $d(n)$ behaves as $\sqrt{n} - \frac{1}{2\sqrt{n}}$ if n large. Although $\lim_{n \rightarrow +\infty} \frac{d(n)}{\sqrt{n}} = 1$, the limit behavior gives no information on the size relation between $d(n)$ and \sqrt{n} . Therefore, we need the following more delicate analysis.

Lemma 8.3. *When $n \geq 15$, we have*

$$d(n) < \sqrt{n}.$$

Proof. We prove the following inequality: For $n \geq 15$,

$$9n^2 - 36n + 96 < \frac{1536 + 1152n^2}{d_0(n)} + \frac{3}{2}d_0(n) < 9n^2 + 96. \quad (8.8)$$

The second inequality of (8.8) is equivalent to the following

$$x^2 - (6n^2 + 64)x + 768n^2 + 1024 < 0. \quad (8.9)$$

Here $x \in (128, 187)$ since $d_0(n) \in (128, 187)$. Next, we show that (8.9) holds. The roots of the equation corresponding to the above inequality are

$$x_1(n) = 3n^2 + 32 - 3n\sqrt{n^2 - 64}, \quad x_2(n) = 3n^2 + 32 + 3n\sqrt{n^2 - 64}.$$

For $n \geq 15$, we have that $x_2(n) \geq x_2(15) \geq 1276 > d_0(n)$. Next we will show that $d_0(n) > x_1(n)$, which implies that (8.9) holds, hence the second inequality of (8.8) holds. Since $n^2 - 64 > (n - 3)^2$ for $n \geq 15$, we have that

$$\begin{aligned} 0 < x_1(n) &= 3n^2 + 32 - 3n\sqrt{n^2 - 64} = \frac{768n^2 + 1024}{3n^2 + 32 + 3n\sqrt{n^2 - 64}} \\ &< \frac{768n^2 + 1024}{3n^2 + 32 + 3n(n - 3)} = \frac{768n^2 + 1024}{6n^2 - 9n + 32}. \end{aligned}$$

To compare $x_1(n)$ with $d_0(n)$, let us compare $x_1^3(n)$ with $d_0^3(n)$. First, we know that

$$\begin{aligned}
& -d_1(n)(6n^2 - 9n + 32)^3 - (768n^2 + 1024)^3 \\
& = 23328n^{12} - 384912n^{11} + 2443608n^{10} - 8266860n^9 - 276048n^8 + 76177584n^7 \\
& \quad - 915397632n^6 + 1095581376n^5 - 4004833536n^4 + 1592960256n^3 \\
& \quad - 2731991040n^2 - 3305373696n + 2038431744 > 0 \text{ if } n \geq 10.
\end{aligned} \tag{8.10}$$

It follows that

$$\begin{aligned}
& \left(-d_1(n)(6n^2 - 9n + 32)^3 - (768n^2 + 1024)^3 \right)^2 - 36^2 d_2(n)(6n^2 - 9n + 32)^6 \\
& = 1358954496 \left(116640n^{17} - 606528n^{16} + 1195560n^{15} + 16771860n^{14} \right. \\
& \quad - 104564844n^{13} + 682366923n^{12} - 1464330096n^{11} + 5142941100n^{10} \\
& \quad - 6506609472n^9 + 15562840464n^8 - 11332244736n^7 + 21360207936n^6 \\
& \quad - 5590593536n^5 + 10574331904n^4 + 4294279168n^3 - 2878341120n^2 \\
& \quad \left. + 3791650816n - 3221225472 \right) \\
& = 1358954496(4320n^{11} - 22464n^{10} + 27000n^9 + 711036n^8 - 4003812n^7 \\
& \quad + 22548513n^6 - 38373440n^5 + 96546304n^4 - 66202112n^3 + 68272128n^2 \\
& \quad + 59244544n - 50331648)(3n^2 + 4)^3 > 0 \text{ if } n \geq 1.
\end{aligned} \tag{8.11}$$

Then combining with (8.10) and (8.11), we get that

$$-d_1(n)(6n^2 - 9n + 32)^3 - (768n^2 + 1024)^3 > 36\sqrt{d_2(n)}(6n^2 - 9n + 32)^3.$$

Hence,

$$-d_1(n) - 36\sqrt{d_2(n)} > \frac{(768n^2 + 1024)^3}{(6n^2 - 9n + 32)^3},$$

that is $d_0^3(n) > x_1^3(n)$, which yields that $d_0(n) > x_1(n)$. Combining with $d_0(n) < x_2(n)$ when $n \geq 15$, we obtain that (8.9). Hence, we get the second inequality of (8.8).

A similar technique can be applied to the first inequality of (8.8), which is equivalent to the following inequality:

$$x^2 - (6n^2 - 24n + 64)x + 768n^2 + 1024 > 0, \tag{8.12}$$

where $x \in (128, 187)$ since $d_0(n) \in (128, 187)$. The roots of the equation corresponding to the above inequality are

$$\begin{aligned}
r_1(n) &= 3n^2 - 12n + 32 - \sqrt{9n^4 - 72n^3 - 432n^2 - 768n}; \\
r_2(n) &= 3n^2 - 12n + 32 + \sqrt{9n^4 - 72n^3 - 432n^2 - 768n}.
\end{aligned}$$

Notice that $9n^4 - 72n^3 - 432n^2 - 768n > 0$ if $n \geq 13$. We next shows that $d_0(n) < r_1(n) < r_2(n)$, hence we get (8.12). For $r_1(n)$, notice that

$$\begin{aligned} 9n^4 - 72n^3 - 432n^2 - 768n &< 9n^4 - 72n^3 - 432n^2 + 2304n + 9216 \\ &= (3n^2 - 12n - 96)^2, \end{aligned}$$

it follows that

$$\begin{aligned} r_1(n) &= \frac{768n^2 + 1024}{3n^2 - 12n + 32 + \sqrt{9n^4 - 72n^3 - 432n^2 - 768n}} \\ &> \frac{768n^2 + 1024}{3n^2 - 12n + 32 + 3n^2 - 12n - 96} = \frac{384n^2 + 512}{3n^2 - 12n - 32} := r_{10}(n). \end{aligned}$$

Notice that $3n^2 - 12n - 32 = 3(n+4)(n-8) > 0$ if $n \geq 9$.

Firstly we observe that

$$\begin{aligned} &-d_1(n)(3n^2 - 12n - 32)^3 - (384n^2 + 512)^3 \\ &= 2916n^{12} - 69984n^{11} + 548208n^{10} - 699840n^9 - 12052800n^8 + 54991872n^7 \\ &\quad - 7831296n^6 - 691006464n^5 - 299151360n^4 + 4048994304n^3 + 3403284480n^2 \\ &\quad - 2821718016n - 3246391296 > 0 \quad \text{if } n \geq 11, \end{aligned} \tag{8.13}$$

and that

$$\begin{aligned} &\left(-d_1(n)(3n^2 - 12n - 32)^3 - (384n^2 + 512)^3\right)^2 - 36^2 d_2(n)(3n^2 - 12n - 32)^6 \\ &= 5435817984 \left(-729n^{16} + 14580n^{15} - 36936n^{14} - 631152n^{13} + 3184272n^{12} \right. \\ &\quad + 6849792n^{11} - 15453504n^{10} - 49876992n^9 - 32256000n^8 - 28111872n^7 \\ &\quad + 268692480n^6 + 613150720n^5 + 898416640n^4 + 1187315712n^3 + 983040000n^2 \\ &\quad \left. + 616562688n + 369098752 \right) \\ &= -5435817984(27n^{10} - 540n^9 + 1260n^8 + 25536n^7 - 123120n^6 - 352960n^5 \\ &\quad + 1058048n^4 + 3124224n^3 - 2383872n^2 - 9633792n - 5767168)(3n^2 + 4)^3 \\ &< 0 \quad \text{if } n \geq 14. \end{aligned} \tag{8.14}$$

Combining with (8.13) and (8.14), we have that

$$-d_1(n)(3n^2 - 12n - 32)^3 - (384n^2 + 512)^3 < 36\sqrt{d_2(n)}(3n^2 - 12n - 32)^3,$$

hence

$$-d_1(n) - 36\sqrt{d_2(n)} < \frac{(384n^2 + 512)^3}{(3n^2 - 12n - 32)^3},$$

that is, $d_0^3(n) < r_{10}^3(n)$, thus $d_0(n) < r_{10}(n) < r_1(n) < r_2(n)$. Therefore, (8.12) holds. This is, the first inequality of (8.8) holds.

Summing up, recall that (8.3), $d(n)$ is well defined and in particular, $d(n) < \sqrt{n}$ for $n \geq 15$. These are direct consequences of (8.8). \square

Next, we will show that, under the case $\frac{n+6}{n-6} < p < p_c(n)$, we have $c_0 > 0, c_1 > 0, c_2 > 0$ simultaneously. Notice that $\frac{n+6}{n-6} < p < p_c(n)$ equivalent to $\min\{0, r_1(n)\} < k < \frac{n-6}{2}$. And $r_1(n)$ is exactly the root of $c_0 = 0$, hence $c_0 > 0$ directly. The condition $\min\{0, r_1(n)\} < k < \frac{n-6}{2}$ is not very applicable, in view of the estimates in Lemma 8.3 and Lemma 8.2 above, we can evaluate c_1, c_2 under the interval $\frac{n-8}{2} - \sqrt{n} < k < \frac{n-8}{2} + \sqrt{n}$. In the two lemmas below, we will follow this idea.

Lemma 8.4. *Under the condition $\frac{n-8}{2} - \sqrt{n} < k < \frac{n-8}{2} + \sqrt{n}$, for $n \geq 36$, we have $c_1 > 0$.*

Proof. Combining with (4.6), we have

$$\begin{aligned} c_1 &= 3k^5 + (54 - 6n)k^4 + (3n^2 - 84n + 372)k^3 + (30n^2 - 408n + 1224)k^2 \\ &\quad + \left(\frac{159}{2}n^2 - 810n + 1917 - \frac{3}{16}n^4 + \frac{3}{2}n^3\right)k + 48n^2 - 480n + 1152. \end{aligned}$$

Set $k = \frac{n-8}{2} + a(n)\sqrt{n}$, where $-1 \leq a(n) \leq 1$. For the simplicity, we denote $a(n)$ by a . Thus,

$$\begin{aligned} c_1 &= 12 + \left(\frac{9}{8} - \frac{3}{4}a^2\right)n^4 + \left(-\frac{3}{2}a^3 + \frac{3}{2}a\right)n^{\frac{7}{2}} + \left(-\frac{39}{4} + \frac{3}{2}a^4 + 3a^2\right)n^3 \\ &\quad + \left(3a^5 - \frac{3}{2}a\right)n^{\frac{5}{2}} + (-6a^4 + 6a^2 + 3)n^2 + (-12a^3 - 18a)n^{\frac{3}{2}} \\ &\quad + \left(24a^2 + \frac{141}{2}\right)n - 3an^{\frac{1}{2}}. \end{aligned}$$

For the case $0 \leq a \leq 1$, since $3a^5 - \frac{3}{2}a \geq -\frac{3}{10^{\frac{5}{4}}}$, we get from the above identity that

$$\begin{aligned} c_1 &\geq 12 + \frac{3}{8}n^4 - \frac{39}{4}n^3 - \frac{3}{10^{\frac{5}{4}}}n^{\frac{5}{2}} - 3n^2 - 30n^{\frac{3}{2}} + \frac{141}{2}n - 3n^{\frac{1}{2}} \\ &= \frac{3}{8}t^8 - \frac{39}{4}t^6 - \frac{3}{10^{\frac{5}{4}}}t^{\frac{5}{2}} - 3t^4 - 30t^3 + \frac{141}{2}t^2 - 3t + 12 \quad (n = t^2) \\ &\geq 0 \quad \text{if } n = t^2 \geq 26.8(t \geq 5.168). \end{aligned}$$

For the case $-1 \leq a \leq 0$, since $\frac{3}{2}(a - a^3) \geq -\frac{\sqrt{3}}{3}$, we have

$$\begin{aligned} c_1 &\geq 12 + \frac{3}{8}n^4 - \frac{\sqrt{3}}{3}n^{\frac{7}{2}} - \frac{39}{4}n^3 - \frac{3}{2}n^{\frac{5}{2}} - 3n^2 - 30n^{\frac{3}{2}} + \frac{141}{2}n \\ &= \frac{3}{8}t^8 - \frac{\sqrt{3}}{3}t^{\frac{7}{2}} - \frac{39}{4}t^6 - \frac{3}{2}t^{\frac{5}{2}} - 3t^4 - 30t^3 + \frac{141}{2}t^2 + 12 \\ &\geq 0 \quad \text{if } n = t^2 \geq 35.98(t \geq 5.999). \end{aligned}$$

□

Lemma 8.5. *Under the condition $\frac{n-8}{2} - \sqrt{n} < k < \frac{n-8}{2} + \sqrt{n}$, for $n \geq 12$, we have*

$$c_2 > 0.$$

Proof. We set $k = \frac{n-8}{2} + a(n)\sqrt{n}$, hence by the assumption we have $-1 \leq a(n) \leq 1$. For simplicity, we denote a for $a(n)$. From (4.6), we have

$$c_2 := -3k^2 + (-36 + 3n)k^2 + (-135 + 27n - \frac{3}{4}n^2)k + 36n - 192. \quad (8.15)$$

Plugging $k = \frac{n-8}{2} + a\sqrt{n}$ in the above expression we get that

$$c_2 = -36 + (\frac{9}{2} - \frac{3}{2}a^2)n^2 + (-3a^3 + 3a)n^{\frac{3}{2}} - \frac{39}{2}n + 9an^{\frac{1}{2}}.$$

If $0 \leq a \leq 1$, we have

$$\begin{aligned} c_2 &\geq -36 + 3n^2 - 3n^{\frac{3}{2}} - \frac{39}{2}n \\ &= 3t^4 - 3t^3 - \frac{39}{2}t^2 - 36 \quad (n = t^2) \\ &\geq 0 \quad \text{if } n = t^2 \geq 10.9025 (t \geq 3.3019). \end{aligned}$$

If $-1 \leq a \leq 0$, we have

$$\begin{aligned} c_{2a} &\geq -36 + 3n^2 - 3n^{\frac{3}{2}} - \frac{39}{2}n - 9n^{\frac{1}{2}} \\ &= 3t^4 - 3t^3 - \frac{39}{2}t^2 - 9t - 36 \quad (n = t^2) \\ &\geq 0 \quad \text{if } n = t^2 \geq 11.8259 (t \geq 3.4388). \end{aligned} \quad (8.16)$$

□

Next, we state a lemma via the numerical analysis of the above arguments.

Lemma 8.6. *Consider the supercritical case $p > \frac{n+6}{n-6}$, i.e., $0 < k < \frac{n-6}{2}$. We have the following facts.*

- (1) *If $0 < k < \frac{n-6}{2}$ and $n \leq 14$, then $c_0, c_1, c_2 > 0$;*
- (2) *If $15 \leq n \leq 50$ and $r_1 < k < \frac{n-6}{2}$, then $c_0, c_1, c_2 > 0$.*

Notice that $k > \min\{r_1 := \frac{n-8}{2} - d(n), 0\}$ is equivalent to $p < p_c(n)$. Combining Lemmas 8.4-8.6 we obtain the proof of Lemma 4.1.

References

- [1] S. Blatt, Monotonicity formulas for extrinsic triharmonic maps and the triharmonic Lane-Emden equations, *preprint* 2015.
- [2] L. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical sobolev growth, *Comm. Pure Appl. Math.*, **42**, no. 3, 271-297, 1989.
- [3] L. Caffarelli and L. Silvestre, An extension problem related to the fractional laplacian, *Comm. Partial Differential Equations*, **32**, no. 7-9, 1245-1260, 2007
- [4] C. Cowan, P. Esposito and N. Ghoussoub, Regularity of extremal solutions in fourth order nonlinear eigenvalue problems on general domains, *DCDS-A* 28 (2010), 1033–1050.
- [5] C. Cowan and N. Ghoussoub, Regularity of semi-stable solutions to fourth order nonlinear eigenvalue problems on general domains, to appear *Cal. Var. PDE* DOI 10.1007/s00526-012-0582-4.
- [6] J. Davila, L. Dupaigne, J. Wei, on the fractional Lane-Emden equation, *Transc. AMS* to appear.
- [7] J. Davila, L. Dupaigne, and K. Wang, J. Wei, A monotonicity formula and a liouville-type Theorem for a fourth order supercritical problem, *Advances in Mathematics*, **258**, 240-285, 2014.
- [8] Lawrence C. Evans. Partial regularity for stationary harmonic maps into spheres. *Arch. Rational Mech. Anal.* 116(2) (1991), 101?13.
- [9] Sun-Yung A. Chang, Lihe Wang, and Paul C. Yang. A regularity theory of biharmonic maps. *Comm. Pure Appl. Math.* 52(9)(1991), 1113?137.
- [10] A. Farina, On the classification of solutions of the Lane-Emden equation on unbounded domains of \mathbb{R}^N , *J. Math. Pures Appl.*, **87**(9), no. 5, 537-561, 2007.
- [11] A. Farina, Liouville-type results for solutions of $-\Delta u = |u|^{p-1}u$ on unbounded domains of \mathbb{R}^N , *C. R. Math. Acad. Sci. Paris*, **341**, no. 7, 415-418, 2005
- [12] A. Farina and A. Ferrero, Existence and stability properties of entire solutions to the polyharmonic equation $(-\Delta)^m u = e^u$ for any $m \geq 1$. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 33 (2016), no. 2, 495?28.
- [13] F. Gazzola, H. C. Grunau, Radial entire solutions of supercritical biharmonic equations, *Math. Ann.*, **334**, 905-936, 2006.
- [14] Yoshikazu Giga and Robert V. Kohn. Asymptotically self-similar blow-up of semilinear heat equations. *Comm. Pure Appl. Math.* 38(3)(1985), 297?19.
- [15] Z.M. Guo and J. Wei, Qualitative properties of entire radial solutions for a biharmonic equation with supercritical nonlinearity, *Proc. American Math. Soc.* 138 (2010), no.11, 3957–3964.
- [16] B. Gidas, J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, *Comm. Pure Appl. Math.* **34**(1981), 525-598.

- [17] B. Gidas, J. Spruck, A priori bounds for positive solutions of nonlinear elliptic equations, *Comm. Partial Differential Equations*, **6**(1981), 883-901.
- [18] M. Fazly, J. Wei, on finite morse index solutions of higher order fractional Lane-Emden equations, *American Journal of Mathematics*, to appear.
- [19] D. D. Joseph, T. S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, *Arch. Rational Mech. Anal.*, **49**, 241-269, 1972/73.
- [20] M. Lazzo and P. G. Schmidt, Radial solutions of a polyharmonic equation with power nonlinearity. *Nonlinear Anal.* 71 (2009), no. 12, e1996e2003.
- [21] C. S. Lin, A classification of solutions of a conformally invariant fourth order equation in \mathbb{R}^N , *Comment. Math. Helv.*, **73**, 206-231, 1998.
- [22] S. Luo, J. Wei and W. Zou, On the equation $p \frac{\Gamma(\frac{n}{2} - \frac{s}{p-1})\Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{n-2s}{2} - \frac{s}{p-1})} = \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2}$, preprint 2016.
- [23] L. Martinazzi, Classification of solutions to higher order Liouville's equation on R^{2m} , *Math. Z.* 263 (2009) 307-29.
- [24] Frank Pacard, Partial regularity for weak solutions of a nonlinear elliptic equation. *Manuscripta Math.* 79(2) (1993), 161-72.
- [25] P. Polacik, P. Quittner, P. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems, I. Elliptic equations and systems, *Duke Math. J.* **139**, no. 3, 555-579, 2007
- [26] P. Quittner and Ph. Souplet, *Superlinear Parabolic Problems: Blow-up, Global Existence and Steady States*, Birkhauser, 2007, ISBN 978-3-7643-8442-5.
- [27] J. Wei, X. Xu, Classification of solutions of higher order conformally invariant equations, *Math. Ann.*, **313**, no. 2, 207-228, 1999.